Week 2 (due Oct. 13)

1. Show that $\mathbb{R}$ is connected.

2. The set of rational numbers $\mathbb{Q}$ is a subset of $\mathbb{R}$; let us regard $\mathbb{Q}$ as a topological space with an induced topology. What are the connected components of $\mathbb{Q}$? Are they open subsets of $\mathbb{Q}$?

3. Let U be a subset of a topological space X. The closure of U is defined as the smallest closed set containing U. Show that the closure of a connected set is connected.

4. Show that the definition of the product topology given in class makes sense. I remind that I defined the base of the topology $\tau_{X\times Y}$ on $X \times Y$ to be made of the sets of the form $U \times V$, where U is an element of some base of τ_X and V is an element of some base of τ_Y. You should prove that this indeed defines an abstract base on $X \times Y$, and that the resulting topology is independent of the choice of bases of τ_X and τ_Y.

5. If Y is a set, then the diagonal subset of $Y \times Y$ consists of all points of the form (y, y), where $y \in Y$.

a. Let Y be a Hausdorff space. Prove that the diagonal subset of $Y \times Y$ is closed.

b. Let f and g be continuous maps from a topological space X to a Hausdorff space Y. Show that the set of points in X where f and g take the same value is closed.

c. A subset U of a topological space X is called dense if its closure is the whole X. Using b, prove that if f and g agree on a dense subset of X, then they agree everywhere on X.