Week 3 (due Oct. 20)

1. Any compact surface can be constructed by taking a polygon and identifying some of its sides. This is called a *classical representation* of a compact surface. E.g. it is well known how to get the cylinder, Moebius strip, and the Klein bottle by identifying some sides of a square. Classical representations are convenient for finding polygonal decompositions of compact surfaces.

(a) Construct a polygonal decomposition of a cylinder and use it to compute its Euler characteristic.

(b) Do the same for the Moebius strip and the Klein bottle.

(c) Do the same for the projective plane.

2. (a) Construct a classical representation of a compact surface obtained by drilling n holes in a sphere.

(b) Construct a classical representation of a compact surface obtained by attaching n handles to a sphere. (Hint: given a classical representation of a surface S, one can find a general recipe for constructing a classical representation for S with an extra handle.)

3. According to the classification theorem for compact surfaces, any nonorientable surface is homeomorphic to a sphere with several cross-caps. Take a Klein bottle and a sphere with two handles and connect them with a cylinder (i.e. take the connected sum). The resulting surface is clearly nonorientable, and therefore is homeomorphic to a sphere with n cross-caps. Find n.

4. (a) Two holes are drilled in a disk. Their boundaries ℓ_1 and ℓ_2 are given the same orientation and glued together in a way which preserves orientation. Show that this is the same as closing each hole with a Moebius strip. (Hint: in order to visualize the gluing, make two cuts so that each cut connects the two holes. Then flip the cut-out piece, so that the boundaries to be glued are next to each, and perform the gluing. Finally, reglue the cuts that you made.)

(b) Cut three holes in a disk. Close one of them (ℓ_1) with a Moebius strip. Give the boundaries of the two remaining holes ℓ_2, ℓ_3 opposite orientations and glue them together (this gives a handle). Show that all this is equivalent to closing each of the three holes with a Moebius strip. In other words, a handle and a cross-cap are equivalent to three cross-caps. (Hint: make a semicircle-shaped cut which begins at ℓ_1, goes around the hole ℓ_2 without touching it, and comes back to ℓ_1. Flip the cut out piece and glue the boundary of ℓ_1 as required. The remaining cut has the shape of a sickle. You also have two circular holes ℓ_2 and ℓ_3. Reglue the cut you made, observing the orientations. Finally glue together the boundaries of ℓ_2 and ℓ_3, keeping in mind the result of part (a)).

5. Let X be a topological space with a marked point x_0. Prove that the composition law of the homotopy classes of based loops in X is associative, even though the composition of the based loops themselves is not. This completes the proof that $\pi_1(X, x_0)$ is indeed a group.

6. (a) Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous functions on a topological space X. Show that $f + g$, $|f|$ and $|g|$ are also continuous.

(b) Let $f_1, \ldots, f_n$ be continuous real-valued functions on a topological space X. Show that $F = max(f_i : i = 1, \ldots, n)$ is also continuous. (Hint: first prove it for two functions. The proof is simplified if one notes that $max(f, g) = \frac{1}{2}(f + g + |f - g|)$.)