Week 4 (due Oct. 27)

1. Let X, Y be topological spaces. Prove that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

2. (a) Write down explicit formulas for the stereographic projections from the north pole p_+ and south pole p_- of a 2-sphere to the plane $\mathbb{R}^2$. If we take these two projections as local systems of coordinates on $S^2 - p_+$ and $S^2 - p_-$, respectively, we get an atlas for S^2 consisting of just two charts. Compute the corresponding transition function and verify that it is a diffeomorphism.

(b) Let z_1, z_2 be homogeneous coordinates on $\mathbb{CP}^1$. Let $\sigma^i, i = 1, 2, 3$, be the usual 2×2 Pauli matrices. Put z_1 and z_2 together into a column-vector z and consider a 3-vector

$$n^i(z) = \frac{z^\dagger \sigma^i z}{z^\dagger z}$$

Show that the map $f : (z_1, z_2) \mapsto \vec{n}(z)$ is a map from $\mathbb{CP}^1$ to a unit sphere S^2, and that this map is one-to-one.

(c) Show that f is a diffeomorphism.

3. Let $(U, x^1, \ldots, x^m)$ be a local coordinate system on M. Show that the tangent vectors $v_i = \frac{\partial}{\partial x^i}, i = 1, \ldots, m$, are linearly independent everywhere on U.

4. Let A be an algebra, and $Der(A)$ be the set of all derivations of A. Show that the commutator operation $[\ ,\ ]$ on $Der(A)$ satisfies the axioms of a Lie algebra.

5. Using the definition of the Lie bracket of vector fields, prove that the following identity holds for any two vector fields X, Y on M and any $f, g \in C^\infty(M)$:

$$[f \cdot X, g \cdot Y] = (f \cdot g) \cdot [X, Y] + f \cdot X(g) \cdot Y - g \cdot Y(f) \cdot X.$$