Week 5 (due Nov. 3)

1. (a) Find the integral curves of the following vector fields on the plane and determine whether these vector fields are complete:

$$D = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ J = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ P_x = \frac{\partial}{\partial x},\ P_y = \frac{\partial}{\partial y}.$$

(b) Compute all the pairwise Lie brackets of these vector fields and show that they form a 4-dimensional real Lie algebra.

2. Introduce complex coordinates $z, \bar{z}$ on the plane and the corresponding partial derivatives. It is convenient to write real vector fields on a plane as real or imaginary parts of complex vector fields. For example, the vector field P_x can be regarded as the real part of the complex vector field $2\frac{\partial}{\partial z}$.

(a) Consider the following vector fields: $L_n = z^{n+1}\frac{\partial}{\partial z}$ and $\bar{L}_n = \bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$, $n \in \mathbb{Z}$. The fields $L_n, \bar{L}_n$ for $n \geq -1$ are well-defined everywhere on the plane, while the fields $L_n, \bar{L}_n$ for $n < -1$ are well-defined away from the origin. Compute the commutators(=Lie brackets) of these vector fields and show that they form a complex Lie algebra, which is a direct sum of two identical Lie algebras. (The summand is called the Witt algebra, or the centerless Virasoro algebra; it plays an important role in string theory).

(b) Show that the vector fields L_1, L_0, L_{-1} form a closed Lie algebra.

(c) Consider the space of all complex matrices of size 2×2 with zero trace. This is a 3-dimensional complex vector space. The matrix commutator is a skew-symmetric bilinear operation on this vector space which satisfies the Jacobi identity, so this vector space can be regarded as a Lie algebra. It is called $sl_2(\mathbb{C})$. Show that it is isomorphic to the Lie algebra in part (b).

(d) We can regard $\mathbb{R}^2$ as one of the two charts needed to cover a 2-sphere. In other words, $\mathbb{R}^2$ can be regarded (via the stereographic projection) as S^2 minus the north pole. The (complex) vector fields L_n, $\bar{L}_n, n \geq -1$, therefore can be regarded as vector fields on S^2 minus the north pole. One can try to extend them smoothly to the whole S^2. Show that such an extension exists only for $n \leq 1$. (Hint: most of these vector fields have a singularity (a pole) at the north pole. To see this, one has to rewrite these vector fields using a local coordinate well-defined in an open neighborhood of the north pole).

3. Let V be a finite-dimensional real vector space with a scalar product denoted $\langle\ ,\ \rangle$. Let $e_i, i = 1, \ldots, n$ be an orthonormal basis for V. Fix the order of these basis vectors (this is called a choice of an orientation on V).

For all $p = 0, \ldots, n$ we consider a linear map

$$\star : \Lambda_p(V) \to \Lambda_{n-p}(V)$$

defined by its action on the basis vectors as follows:

$$\star(1) = e_1 \wedge e_2 \wedge \ldots \wedge e_n, \quad \star(e_1 \wedge e_2 \wedge \ldots \wedge e_n) = 1;$$

if $i_1, i_2, \ldots, i_n$ is obtained from $1, 2, \ldots, n$ by a permutation π, then

$$\star\,(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) = (-1)^{\pi} e_{i_{p+1}} \wedge e_{i_{p+2}} \wedge \ldots \wedge e_{i_n}.$$

This linear map is called the Hodge star operator.

(a) If we act by $\star$ twice, we get back to $\Lambda_p(V)$. Show that $\star\star$ regarded as an operator from $\Lambda_p(V)$ to itself is proportional to the identity operator; more precisely, show that

$$\star\star\,(\alpha) = (-1)^{p(n-p)}\alpha, \ \forall \alpha \in \Lambda_p(V).$$

(b) The definition of the Hodge star operation seems to depend on the choice of an orthonormal basis. Let $f_1, \ldots, f_n$ be another orthonormal basis for V. It is related to the old basis $e_1, \ldots, e_n$ by an orthogonal matrix A^i_j:

$$f_i = A^j_i e_j$$

(summation over repeated indices is implied, as usual). The determinant of A can be $+1$ or -1. If $\det A = 1$, the new basis defines the same orientation on V as the old one; if $\det A = -1$, it defines the opposite orientation (by the definition of orientation). We will denote the "old" Hodge star by $\star_e$ and the new Hodge star by $\star_f$. Show that $\star_e = \star_f$ if $\det A = 1$ and $\star_e = -\star_f$ if $\det A = -1$. This shows that the Hodge star operation actually depends only on the choice of scalar product and the orientation on V.