Week 6 (due Nov. 10)

1. Prove that the Cartan formula for the exterior derivative of $\omega \in \Omega^k(M)$ (see Morita, Theorem 2.9) indeed defines a differential form on M. (Hint: it is sufficient to show that the expression for $d\omega(X_1, \ldots, X_k)$ is linear with respect to multiplication of arguments $X_1, \ldots, X_k$ by arbitrary smooth functions on M.)

2. Let ω be a form on $\mathbb{R}^{2n}$ defined by

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \ldots + dx^{2n-1} \wedge dx^{2n}.$$

It is called the standard symplectic form on $\mathbb{R}^{2n}$. Compute the form ω^n (the n^{th} exterior power of ω).

3. This is a linear algebra problem.

(a) Let V be a vector space. Show that elements $v_1, \ldots, v_k \in V$ are linearly independent iff $v_1 \wedge \ldots \wedge v_k \neq 0$.

(b) Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ be two sets of linearly independent vectors in V. Show that they span the same subspace of V iff $v_1 \wedge \ldots \wedge v_k = cw_1 \wedge \ldots \wedge w_k$, where $c \in \mathbb{R}$ is a non-zero constant.

(c) Let A be a linear operator on V. It induces a linear transformation on $\Lambda^n(V)$, where $n = \dim V$. But since $\Lambda^n(V)$ is a one-dimensional vector space, the induced linear transformation must be simply a multiplication by a number. Show that this number is equal to $\det A$.

4. (a) Problem 2.7 in Morita (p. 94).

(b) Specialize the $n-1$-form ω considered in part (a) to the case $n = 3$ and rewrite it using spherical coordinates r, θ, ϕ.

(c) Consider vector fields on $\mathbb{R}^3$:

$$J_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \; J_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \; J_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Compute their commutators and show that they form a Lie algebra. Compute the Lie derivative of the form ω from part (b) along the vector fields J_x, J_y, J_z.

(d) Compute the Lie derivative of the form ω from part (a) (for general n) along the vector field

$$\sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}.$$