Week 7 (due Nov. 17)

1. The Lie derivative L_X of a tensor field V along a vector field X is completely determined by the following properties:

(i) $L_X(V \otimes W) = L_X V \otimes W + V \otimes L_X W$ for any two tensor fields V and W.

(ii) $L_X(Y) = [X, Y]$ and $L_X(f) = X(f)$ for any vector fields X, Y and any function f.

(iii) $[L_X, d] = 0$, where d is the exterior derivative.

(a) Using these properties, compute the Lie derivative of a 1-form in local coordinates and verify the identity $L_X = i_X \circ d + d \circ i_X$.

(b) Let J be a tensor field of type (1,1), i.e. in local coordinates it has the form

$$J = J^i_j(x) \frac{\partial}{\partial x^i} \otimes dx^j.$$

Let $X = v^i \frac{\partial}{\partial x^i}$ be a vector field. Compute the expression in local coordinates for the tensor field $L_X J$.

2. If P is a section of $\Lambda^k(TM)$, we will set $|P| = k$ and say that P has degree k. The Schouten-Nijenhuis bracket $[\ ,\ ]$ on $\Lambda^*(TM)$ is completely determined by the following properties:

(i) $[aP + bQ, R] = a[P, R] + b[Q, R]$ for any $a, b \in \mathbb{R}$.

(ii) $|[P, Q]| = |P| + |Q| - 1$.

(iii) $[P, Q] = -(-1)^{(|P|-1)(|Q|-1)}[Q, P]$.

(iv) $[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(|P|-1)|Q|} Q \wedge [P, R]$.

(v) If X, Y are vector fields, and f is a function, then $[X, Y]$ is the Lie bracket of X and Y, and $[X, f] = X(f)$.

(a) Let P, Q be bi-vector fields on M, i.e. sections of $\Lambda^2(TM)$. Compute $[P, Q]$ in local coordinates.

(b) If X is vector field, and P is a polyvector field, show that $L_X P = [X, P]$, where the bracket is the Schouten-Nijenhuis bracket.

3. Recall that for any algebra A its derivations $Der(A)$ form a Lie algebra. We want to use this fact to invent the definition of a Lie superalgebra. Let A be a superalgebra, i.e. a vector space A with a decomposition $A = A_0 \oplus A_1$ and a bilinear multiplication satisfying

$$A_i \cdot A_j \subset A_{i+j},$$

where the indices i, j take values in $\mathbb{Z}_2$. If $a \in A_i$, we will say that a has degree $i \in \mathbb{Z}_2$, and denote it $|a|$. A derivation of degree $p \in \mathbb{Z}_2$ is linear map

$D : A \to A$ such that $D(A_i) \subset A_{i+p}$ and the super-Leibniz rule is satisfied:

$$D(a \cdot b) = D(a) \cdot b + (-1)^{p|a|} a \cdot D(b).$$

The degree of the a derivation D will be denoted $|D|$. We denote by $Der(A)$ the set of all derivations of a superalgebra A; $Der(A)$ itself is a supervector space, since we can decompose it into derivations of degree 0 and derivations of degree 1.

(a) Let us define the super-Lie bracket on $Der(A)$ by

$$[D_1, D_2] = D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1.$$

Why do we need to insert the peculiar power of (-1) in this definition? Could one simply drop it? Determine the symmetry properties of this bracket, i.e. express $[D_2, D_1]$ through $[D_1, D_2]$.

(b) Find the super-version of the Jacobi identity satisfied by the super-Lie bracket on $Der(A)$.