Week 8 (due Nov. 24)

(**NOTE**: no class on Monday Nov. 22)

1. (a) Consider a 2-form on $\mathbb{R}^3 - \{0\}$:

$$\sigma = \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}.$$

Rewrite it in spherical coordinates r, θ, ϕ.

(b) Prove that the form σ is closed, but not exact. This shows that the second de Rham cohomology of $\mathbb{R}^3$ with a point removed is nontrivial.

(c) Compute the 1-form $\star\sigma$ and the corresponding vector field $\widetilde{\star\sigma}$. Show that σ satisfies $d \star \sigma = 0$ (the sourceless Maxwell equation). What does this equation look like in terms of the vector field $\widetilde{\star\sigma}$?

(d) Let α be a 1-form on $\mathbb{R}^n - \{0\}$ given by

$$\alpha = \frac{\sum_{i=1}^n x_i dx_i}{r^n},$$

where $r = (x_1^2 + \ldots + x_n^2)^{1/2}$. Show that $\star\alpha$ is closed, but not exact.

2. Compute the action of the Hodge star operator in a coordinate (as opposed to orthonormal basis). That is, find how $\star$ acts on

$$dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

and express the result in terms of differentials dx^i, the components of the metric tensor g_{ij} and the completely anti-symmetric symbol $\epsilon_{i_1 \ldots i_n}$.

3. Let X be a vector field on a Riemannian manifold (M, g_{ij}) which in local coordinates looks like

$$X = V^i(x) \frac{\partial}{\partial x^i}.$$

Show that the divergence of X (defined as $\star d \star \tilde{X}$) is given in local coordinates by the following expression:

$$\mathrm{div} X = g^{-1/2} \frac{\partial \left(g^{1/2} V^k \right)}{\partial x^k},$$

where $g = \det g_{ij}$.