# Week 9 (due Dec. 1)

1. (a) Consider a unit sphere in $\mathbb{R}^3$. Since it is a submanifold of $\mathbb{R}^3$, we can restrict the usual metric $dx \otimes dx + dy \otimes dy + dz \otimes dz$ to $S^2$. Compute the resulting metric tensor on $S^2$ in spherical coordinates $(\theta, \phi)$. What is the action of the Hodge star operator on $d\theta$ and $d\phi$?

(b) Let $\delta = \star d \star$. Consider the equation $\delta\alpha = 0$, where $\alpha$ is a 2-form on $S^2$. Find the general smooth solution of this equation.

(c) Let $\phi : S^2 \to S^2$ be the antipodal map, i.e. the map which sends a point on $S^2$ to its opposite. Let $\alpha \in \Omega^2(S^2)$ be a solution of the equation $\delta\alpha = 0$ (see part (b)). What is the relation between $\alpha$ and $\phi^*\alpha$?

2. A Weyl transformation of a Riemannian metric is a rescaling of the Riemannian metric by a positive smooth function on $M$.

(a) Determine the transformation law of the Hodge star operator under a Weyl transformation.

(b) Consider the standard action for the $p$-form field in $n$-dimensions:

$$S[\omega] = \int_M d\omega \wedge *d\omega, \qquad \omega \in \Omega^p(M)$$

For which values of $p$ and $n$ is this action invariant under Weyl transformations? (Note: the field $\omega$ may transform non-trivially under Weyl transformations, and your goal is to determine its transformation law so that the action is Weyl-invariant.)

(c) The equation of motion for the self-dual $p$-form field in $2(p+1)$ dimensions is

$$d\omega = *d\omega.$$

Is it Weyl-invariant?

3. In $2p+1$ dimensions, one can write down an action for a p-form $\omega$ which is independent of metric on space-time. This action is known as the (generalized) Chern-Simons action:

$$S = \int_M \omega \wedge d\omega.$$

This field theory is called topological, because it depends only on the topology (more precisely, smooth structure) of $M$.

(a) Consider a transformation $\omega \to \omega + d\alpha$, where $\alpha$ is an arbitrary $p-1$-form. Show that the action is invariant under these "gauge" transformations.

If two $p$-forms $\omega_1$ and $\omega$ are related by a gauge transformations, we will say that they are gauge-equivalent. A $p$-form which is gauge equivalent to zero is said to be pure gauge.

(b) Derive the equations of motion following from the Chern-Simons action. Show that the space of solutions of equations of motion modulo those which are pure gauge is isomorphic to the de Rham cohomology group $H^p(M)$. (In particular, for compact $M$ it is finite-dimensional. This illustrates a general fact that topological field theories typically have a finite number of degrees of freedom. Upon quantization, this means that their Hilbert space is finite-dimensional, unlike for ordinary quantum field theories like QED).