Week 2 (due April 13)

1. Consider the group $G = SU(2)$. A natural basis in its Lie algebra $\mathfrak{g}$ is given by $i\sigma_k, k = 1, 2, 3$, where σ_k are the Pauli matrices. Compute the basis of left-invariant 1-forms on G dual to this basis of $\mathfrak{g}$. Use the following standard parametrization of $SU(2)$ by "Euler angles" θ, ϕ, ψ:

$$g(\theta, \phi, \psi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}.$$

The ranges for the Euler angles are

$$0 < \theta < \pi, \ 0 \leq \phi < 2\pi, \ -2\pi \leq \psi < 2\pi.$$

2. There is a definition of an associated bundle which is equivalent to the one explained in class but does not use any choice of an open cover. Let (E, B, π, G) be a principal G-bundle with base B. Let F be a topological space on which G acts continuously from the left. On the product $E \times F$ we have a right action of G defined as follows:

$$(p, x)g = (p \cdot g, g^{-1}x), \quad \forall p \in E, \ \forall x \in F, \ \forall g \in G.$$

Here $p \cdot g$ denotes the result of right action of g on the point $p \in E$ (I remind that the structure group G acts on E from the right). Consider the quotient $E' = (E \times F)/G$, and define the projection $\pi' : E' \to B$ in the obvious way (the equivalence class of (p, x) is mapped to $\pi(p)$). Show that (E', B, π', F) is a fiber bundle with typical fiber F. It is an associated bundle (meaning "associated to the principal G-bundle E via the action of G on F").