Week 4 (due April 27)

1. Let V be a vector space with a Euclidean metric. Let $e_1, \dots, e_n$ be an orthonormal basis in V, and let $\omega = e_1 \dots e_n \in Cl(V)$. Show that if V is even-dimensional, the center of $Cl(V)$ consists of scalars, while if V is odd-dimensional, the center is two-dimensional and spanned by 1 and ω.

2. (a) Let X be a sphere with n handles. Let E be a principal bundle over X with structure group $SU(N)/\mathbb{Z}_N$ and vanishing 't Hooft magnetic flux. That is, E can be lifted to a principal $SU(N)$ bundle. How many inequivalent lifts are there?

(b) The same, but with X being a sphere with n cross-caps.

3. Let $Cl(k)$ denote the Clifford algebra for a k-dimensional Euclidean space. Show that the even part of $Cl(k)$ is isomorphic (as an algebra) to $Cl(k-1)$.

4. Show that $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$. Hint: note that $SO(n)/SO(n-1) = S^{n-1}$. Thus $SO(n)$ is a fibration over S^{n-1} with fiber $SO(n-1)$. Then use the exact sequence of a fibration for homotopy groups.