Week 5 (due May 4)

1. Consider an infinitesimal orthogonal transformation of $\mathbb{R}^n$ which looks like

$$\delta x^\mu = \alpha^\mu_\nu x^\nu,$$

where the real matrix α^μ_ν parametrizes the Lie algebra of $SO(n)$. Orthogonality implies that $\alpha_{\mu\nu}$ is antisymmetric. Using the definition of $Spin(n)$ given in class, determine the the generator corresponding to $\alpha_{\mu\nu}$ in the spinor representation. (Hint: an infinitesimal rotation can be obtained as a composition of two consecutive reflections in the hyperplanes orthogonal to two slightly different vectors in $\mathbb{R}^n$. Since we know how reflections are represented in the spinor representation, we can also deduce how infinitesimal rotations are represented.)

2. (a) Show that the Spin group for the 3d Minkowski space $\mathbb{R}^{2,1}$ is isomorphic to $SL(2,\mathbb{R})$. Construct explicitly the isomorphism from $SL(2,\mathbb{R})$ to $SO(2,1)$ and show that its kernel is $\mathbb{Z}_2$.

(b) Is $SL(2,\mathbb{R})$ simply-connected?

3. Let M be the 2-sphere with the standard Riemannian metric. We can use the standard cover of M by two affine charts to trivialize any vector bundle over M.

(a) Write down the transition functions and the connection 1-forms for the two spinor bundles on M.

(b) Find all solutions of the massless Dirac equation $\not{D}\psi = 0$ on M, where ψ is a smooth section of the direct sum of the two spinor bundles.