Week 6 (due May 11)

1. Consider the Pontryagin genus corresponding to the function $cosh\frac{\sqrt{y}}{2}$. Compute its expression in terms of Pontryagin classes $p_1, p_2, \ldots$ up to (and including) terms of degree 8.

2. Let X be a 2-torus regarded as the quotient of the plane by the lattice generated by unit translations of x and y. Consider a complex line bundle L on X with a Hermitian metric and unitary connection ∇. By going to the universal cover of X, we can regard the connection 1-form of ∇ as an ordinary 1-form on $\mathbb{R}^2$. Consider the following 1-form on $\mathbb{R}^2$:

$$\omega = 2\pi c\, xdy,$$

where c is a constant. (We use the physical convention where the connection 1-form of a unitary connection is a 1-form with values in Hermitian matrices, rather than anti-Hermitian ones.)

(a) Show that this can be regarded as coming from a connection 1-form on L if and only if c is an integer. Show that in this case c is simply the value of the integral of the first Chern class of L over X.

(b) We pull back L to $\mathbb{R}^2$ and get a trivial line bundle on $\mathbb{R}^2$ (since any vector bundle on $\mathbb{R}^2$ is trivial). Thus the pull-back of a section of L is simply a function on $\mathbb{R}^2$ satisfying some quasiperiodicity conditions. Write down these quasiperiodicity conditions assuming that the connection 1-form ω is as above.

(c) Let $S = S_+ \oplus S_-$ be the spinor bundle over X (equipped with the standard Riemannian metric). This is a trivial vector bundle of rank two. Consider the Clifford bundle $S \otimes L$. Write down the massless Dirac equation for this Clifford bundle and find all solutions of the equations

$$\not{D}_\pm \psi = 0$$

Verify that the index theorem is satisfied.