Week 8 (due May 25)

1. The tautological line bundle over $\mathbb{CP}^n$ is denoted $\mathcal{O}(-1)$. Its dual is denoted $\mathcal{O}(1)$ and has a special name: the hyperplane section bundle. Let $\mathcal{O}(k)$ be the k^{th} tensor power of $\mathcal{O}(1)$.

(a) Show that the space of global holomorphic sections of $\mathcal{O}(k)$ is isomorphic to the space of degree-k homogeneous polynomials of $n+1$ variables $z_0, \ldots, z_n$.

(b) Let $z_0, \ldots, z_n$ be homogeneous coordinates on $\mathbb{CP}^n$. An equation $a_0 z_0 + \ldots + a_n z_n = 0$, where $a_0, \ldots, a_n$ are complex numbers, defines a complex submanifold in $\mathbb{CP}^n$ isomorphic to $\mathbb{CP}^{n-1}$. Show that the restriction of the line bundle $\mathcal{O}(k)$ on $\mathbb{CP}^n$ to this submanifold is isomorphic to $\mathcal{O}(k)$ on $\mathbb{CP}^{n-1}$. Deduce from this the value of the first Chern class of $\mathcal{O}(k)$ on a homology class represented by a $\mathbb{CP}^1$ embedded in $\mathbb{CP}^n$ (I remind that $H_2(\mathbb{CP}^n, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and is generated by this $\mathbb{CP}^1$).

2. Let M be a complex manifold of complex dimension n. The n^{th} exterior power of the holomorphic cotangent bundle of M is a line bundle denoted K_M and called the canonical line bundle of M.

(a) Show that $c_1(K_M) = -c_1(TM)$.

(b) Compute the transition functions for K_M in the case when $M = \mathbb{CP}^n$. Show that K_M is isomorphic to $\mathcal{O}(-n)$.