# Week 10 (due March 16, sharp)

1. Let $M$ be a manifold equipped with a Riemannian metric $g$. Let $\nabla$ be a connection on $TM$, not necessarily orthogonal or torsion-free. Let $T$ be the torsion tensor of $\nabla$. Also, define a 3-index tensor $B$ with components $B_{ijk}$ by

$$B_{ijk} = \nabla_i g_{jk}.$$

Express the connection 1-forms $\omega^i_{jk}$ for $\nabla$ in terms of the Levi-Civita connection $\Gamma^i_{jk}$ and the tensors $T$ and $B$.

2. Let $(M, g)$ be a Riemannian manifold of dimension 4, and let $R$ be the curvature 2-form of the Levi-Civita connection $\nabla$. $g$ is called self-dual if $R$ is a self-dual 2-form, i.e.

$$R = \star R$$

Show that any self-dual metric satisfies the Ricci-flatness condition (i.e. satisfies the vacuum Einstein equations).

3. Consider a metric on $\mathbb{R}^2$ of the form

$$ds^2 = \frac{dx^2 + dy^2}{(a^2 + x^2 + y^2)^2},$$

where $a$ is a constant. Take the following obvious orthonormal basis of 1-forms:

$$\theta^1 = \frac{dx}{a^2 + x^2 + y^2}, \quad \theta^2 = \frac{dy}{a^2 + x^2 + y^2}.$$

Compute the Levi-Civita connection in this basis, its curvature 2-form $R$, and the integral of the 2-form $\mathrm{Pf}(R)$ over $\mathbb{R}^2$.

4. (a) A vector field $\xi$ on a Riemannian manifold $(M, g)$ is called a Killing vector field if the Lie derivative of the metric tensor along $\xi$ vanishes. Show that this condition is equivalent to the following one:

$$(\nabla_i \xi)_j + (\nabla_j \xi)_i = 0,$$

where $\nabla$ is the Levi-Civita connection.

(b) Let $T_{ik}$ be a symmetric tensor whose covariant divergence (with respect to the Levi-Civita connection) vanishes:

$$\nabla_i T^i_k = 0.$$

Let $\xi$ be a Killing vector field on $M$. Show that the 1-form $J_i = \xi^k T_{ik}$ is coclosed:

$$d \star J = 0.$$

(This means that the vector field $J^i$ has vanishing divergence, i.e. it is a conserved current).