Week 3 (due Jan. 26)

1. (a) Prove that for any two topological spaces X, Y $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$.

(b) A section of a fiber bundle $p : E \to B$ is a continuous map $s : B \to E$ such that $p \circ s = id$. Prove that the Hopf bundle $S^3 \to S^2$ does not have any sections.

(c) Let $p : E \to B$ be a fiber bundle with typical fiber F. Suppose it has a section. Show that $\pi_n(E) = \pi_n(B) \times \pi_n(F)$, for any n. (This generalizes part (a)).

(d) Prove that if a covering map has a section, then the covering is isomorphic to a trivial one (i.e. the one for which the covering map is an identity map).

2. Let X be $\mathbb{R}^m$ with k points removed. Compute $\pi_n(X)$ for $n < m - 1$. (Hint: show that X is homotopic to a CW-complex which only has cells of dimensions 0 and $m - 1$.)

3. Let X be a topological space, A be its subspace, and $x_0 \in A$ be a marked point. Such a triple (X, A, x_0) is called a *pointed pair*. It is common to suppress x_0 and write the pointed pair simply as (X, A). Let n be a positive integer. *A relative n-dimensional spheroid in* (X, A) is a map $f : I^n \to X$ such that $f(\partial I^n) \subset A$, and $f(I^n - I^{n-1}) = x_0$. Here I^{n-1} is identified with one particular open face of the cube I^n (say, the one given by the equation $x_n = 1$ and inequalities $0 < x_i < 1$ for $i = 1, \ldots, n-1$.) A relative homotopy of relative spheroids is defined in an obvious fashion.

(a) Define a mutliplication on the set of homotopy equivalence classes of relative n-spheroids for $n \geq 2$, so that this set becomes a group. The corresponding group is called *the homotopy group of the pointed pair* (X, A, x_0), or *a relative homotopy group* and denoted $\pi_n(X, A, x_0)$. (Hint: the formulas are the same as for ordinary homotopy groups).

(b) Show that if the image of a relative spheroid belongs to A, then it is relative-homotopic to the constant spheroid (the one which maps everything to x_0).

(c) Ordinary homotopy groups (also known as absolute homotopy groups) are commutative for $n > 1$. What is the analogue of this statement for relative homotopy groups? Justify your answer.