Week 4 (due Feb. 2)

1. Prove the following statements about complexes of abelian groups.

a. If a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

is an exact sequence, then f is surjective iff h is injective.

b. If a sequence

$$\ldots \to C_{n+1} \to A_n \xrightarrow{h_n} B_n \to C_n \to A_{n-1} \xrightarrow{h_{n-1}} B_{n-1} \to \ldots$$

is exact, and every third arrow $h_n : A_n \to B_n$ is an isomorphism, then $C_n = 0$ for all n.

c. If $0 \to A \to B \to C \to 0$ is a short exact sequence, then $\mathrm{rank} B = \mathrm{rank} A + \mathrm{rank} C$.

d. If $0 \to A_n \to A_{n-1} \to \cdots \to A_0 \to 0$ is an exact sequence of finitely-generated abelian groups, then

$$\sum_{i=0}^{n} (-1)^i \mathrm{rank} A_i = 0.$$

e. Let $C.$ be a bounded complex of finitely-generated abelian groups. By definition, the i-th Betti number of $C.$ is $b_i = \mathrm{rank}\ H_i(C.)$. Show that

$$\sum_i (-1)^i \mathrm{rank}\ C_i = \sum_i (-1)^i b_i.$$

This statement is known as the algebraic Euler-Poincare theorem.

2. Show that if X is path-connected, and $A \subset X$ is non-empty, then $H_0(X, A) = 0$.

3. a. Let X, Y be topological spaces, and A, B be their subspaces. Let $f : X \to Y$ be a map such that $f(A) \subset B$ and f_* induces isomorphism of $H_i(X), H_i(A)$ and $H_i(Y), H_i(B)$ for all i. Show that f_* induces an isomorphism of $H_i(X, A)$ and $H_i(Y, B)$ for all i. (Hint: make use of the so-called "five-lemma" on page 184 of Massey).

b. Let D^n be a closed n-dimensional ball, and p be any point in its interior. Show that $H_i(D^n, D^n - p) = H_i(S^n)$ for all i. (Hint: apply (a) to a suitable map $f : D^n \to D^n$.)