Week 7 (due Feb. 23)

1.Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of a topological space X whose elements are labeled by sets A and B, respectively. $\mathcal{V}$ is called a refinement of $\mathcal{U}$ if there is a map $\phi : B \to A$ such that $\forall b \in B$ we have an inclusion $V_b \subset U_{\phi(b)}$. Let $C^q(\mathcal{U}, G)$ and $C^q(\mathcal{V}, G)$ be groups of Cech q-cochains with values in an abelian group G.If $\mathcal{V}$ is a refinement of $\mathcal{U}$, we define a homomorphism $\phi^\sharp : C^q(\mathcal{U}, G) \to C^q(\mathcal{V}, G)$ by letting

$$(\phi^\sharp \omega)_{b_0 \ldots b_q} = \omega_{\phi(b_0) \ldots \phi(b_q)}, \quad \forall \omega \in C^q(\mathcal{U}, G).$$

(a) Show that $\phi^\sharp$ is a chain map (i.e. that it commutes with the Cech coboundary operator), and therefore induces a map from $H^q(\mathcal{U}, G)$ to $H^q(\mathcal{V}, G)$.

(b) Show that if ϕ and ψ are two different "refinement maps" for the same pair of covers $\mathcal{U}, \mathcal{V}$, then the corresponding chain maps $\phi^\sharp$ and $\psi^\sharp$ are chain homotopic and therefore induce identical maps on cohomology groups.

(These two facts are used to define Cech cohomology without using good covers. One considers the disjoint union of Cech cohomology group $H^q(\mathcal{U}, G)$ over all possible covers $\mathcal{U}$ and then identifies elements in this disjoint union if they get identified under some common refinement. This construction is called "direct limit over all open covers" and is explained in detail in the book by Bott and Tu, for example.)

2. Let E be a real rank-1 vector bundle over a space X admitting a good cover. Prove that if the 1st Stiefel-Whitney class of E vanishes, then E has an oriented trivialization (i.e. E is orientable). The argument was sketched in class; in this problem you are required to give a precise proof.

3. (a) Prob. 5.1 in Morita.

(b) Prob. 5.2 in Morita.

(c) Prob. 5.4 in Morita. Also, show that despite the result of this problem, the tangent bundle of any even-dimensional sphere is nontrivial.

4. Let the connection 1-forms on a vector bundle E be

$$\omega^i_j = \omega^i_{j\mu} dx^\mu.$$

Let is denote by A_μ a matrix function on the local chart whose (i, j) matrix element is $\omega^i_{j\mu}$. Also, if the curvature 2-form is

$$\Omega^i_j = \frac{1}{2} \Omega^i_{j\mu\nu} dx^\mu \wedge dx^\nu,$$

let us denote by $F_{\mu\nu}$ a matrix function whose (i,j) matrix element is $\Omega^i_{j\mu\nu}$. Using the formula

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j,$$

express $F_{\mu\nu}$ through the functions A_μ.

5. Let $E^\vee$ be the bundle dual to E. I.e. its fiber at a point p is the space of linear functions on the fiber of E at p. Given a connection ∇_X on E, we can define a connection $\nabla^\vee_X$ on $E^\vee$ as follows. If $\eta \in \Gamma(E^\vee)$ and $s \in \Gamma(E)$, we let

$$(\nabla^\vee_X \eta)(s) = X(\eta(s)) - \eta(\nabla_X s).$$

(a) Show that $\nabla^\vee_X$ is indeed a connection, i.e. that is satisfies the required properties. Will this be true if one removes the minus sign from the above definition?

(b) Given a local trivialization e_i of E, we have a dual local trivialization f^i for $E^\vee$ defined by $f^i(e_j) = \delta^i_j$. Suppose the connection 1-forms for ∇_X in a particular trivialization are ω^i_j. What are the connection 1-forms for $\nabla^\vee_X$ in the dual trivialization? Also, express the curvature 2-form of $\nabla^\vee_X$ in terms of that for ∇_X.

6. Let $End(E)$ be the vector bundle whose fiber at a point p is the space of linear maps from the fiber of E to itself. It is known as the endomorphism bundle of E. Given a connection on E we can define a connection on $End(E)$ as follows: for any $r \in \Gamma(End(E))$ and any $s \in \Gamma(E)$ we let

$$(\nabla_X(r))(s) = \nabla_X(r(s)) - r(\nabla_X s).$$

Here $r(s)$, for example denotes the section of E obtained by applying r pointwise to s.

(a) Show that this is indeed a connection.

(b) Given a trivialization of E, one gets a natural trivialization of $End(E)$. Express the connection 1-forms and curvature 2-form for $End(E)$ in terms of those for E.