Week 9 (due March 9)

1. (a) Derive the Yang-Mills equations of motion by varying the Yang-Mills action

$$S_{YM} = \frac{1}{g^2} \int_M \mathrm{Tr}(F \wedge \star F),$$

where g is a normalization constant. (In this problem one does not have to assume anything about the dimension of M).

(b) Now let us consider the case $\dim M = 3$ and add to S_{YM} the Chern-Simons term

$$S_{CS} = k \int_M \mathrm{Tr}(AdA + \frac{2}{3}A^3),$$

where k is a number. Derive the equations of motion in this case and show that they are gauge-invariant. What do the equations of motion become in the limit $g \to \infty$? Can you find the general solution in this case?

(c) Compute the change in the Chern-Simons action under gauge transformations. In particular, show that the change is an integral of a closed 3-form.

(d) Now specialize (b) to the case when the vector bundle has rank 1. In this case the curvature F is an ordinary 2-form. Show that the equations of motion imply that F satisfies the "Klein-Gordon" equation

$$(\Delta + a)F = 0,$$

where Δ is the Laplace operator on forms, and a is a constant. Express a in terms of g^2 and k. (This excercise shows that modifying Maxwell theory in 3d with a Chern-Simons term makes photons massive without destroying gauge-invariance. Such a modification is impossible in higher dimensions).

2. Consider the 6-form $s_3 = \mathrm{Tr}\Omega^3$, where Ω is a curvature 2-form of a connection ∇. Write this 6-form locally as as the exterior derivative of a 5-form (the latter is called the Chern-Simons 5-form).