Week 8 (due March 2)

Reading: Rubakov.

1. A nonlinear sigma-model in d dimensions is a QFT where the basic field is a map $\phi : \mathbb{R}^d \to M$, where M is a fixed Riemannian manifold, and the action is

$$S = \frac{1}{2} \int d^d x g_{ij}(\phi(x)) \partial_\mu(x) \phi^i \partial^\mu \phi^j(x).$$

M is called the target space of the sigma-model. For example, the chiral effective action to leading order in derivatives (i.e. keeping only the term with 2 derivatives) is a nonlinear sigma-model in 4 dimensions with the target space $M = SU(N_f)$.

(1) Show that a nonlinear sigma-model can have static solutions with a finite energy only for $d = 3$.

(2) Consider the case $d = 3$ and $M = S^2$ with the usual round metric. Using $\pi_2(S^2) = \mathbb{Z}$, show that the model has a topological charge Q taking values in $\mathbb{Z}$. Write down the conserved current corresponding to this charge.

(3) Find the static soliton in this model with $Q = 1$. Write it using two common parameterizations of S^2: as a unit sphere in $\mathbb{R}^3$ and as a complex plane with an extra point at infinity. In the latter case the soliton is thought of as a map $\mathbb{R}^2 \to \mathbb{C}$.

(4) Show that for $d = 3$ and $M = S^2$ the energy of an arbitrary static configuration with charge Q is bounded from below by $a|Q|$, where a is a positive numerical constant. Find a. Note that this is the form of a Bogomolny bound, and you can prove it in the same way as for instantons.

(5) Find all static solutions saturating the Bogomolny bound from part (4) using the "complex" parameterization of S^2 (for arbitrary Q). Don't be scared! Unlike in the case of instantons in 4d YM theory, this is very easy!