Week 2 (due April 21)

Reading: Becker-Becker-Schwarz 4.6-4.7, Polchinski 4.2-4.3 (vol. 1), Polchinski 10.1-10.6 , see also Green-Schwarz-Witten vol. 1.

1. Consider an anti-symmetric tensor field with p indices, $A_{i_1 \dots i_p}$, or equivalently a p-form $A = \frac{1}{p!} A_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p}$, with an action

$$S = \int_{\mathbb{R}^D} dA \wedge *dA,$$

where $*$ is the Hodge star. This theory has a $(p-1)$-form gauge invariance

$$A \mapsto A + d\lambda,$$

where λ is an arbitrary $(p-1)$ form field. For $p = 1$ this theory describes a massless vector particle (photon) in D dimensions, and it is well-known that a photon has $(D-2)$ polarization states.

(a) Define an analog of the Lorenz gauge for general p and write the mode expansion for the field A in this gauge. Show that for a fixed light-like D-momentum k an analog of the polarization vector is a p-form ζ satisfying $k_i \zeta^{ij_1 \dots j_{p-1}} = 0$. Show that there is a residual gauge symmetry and determine how ζ transforms under this symmetry.

(b) Show that physical polarization states transform as a rank-p anti-symmetric tensor of the "little group" $SO(D-2)$. In particular, show that for $D = 10$, the particle corresponding to a 2-form field has 28 polarizations states, and the particle corresponding to a 3-form field has 35 polarization states.

2. Consider $2N$ free left-moving real fermions ψ^i, $i = 1, \dots, 2N$, with the standard OPE

$$\psi^i(z)\psi^j(w) \sim \frac{\delta^{ij}}{z-w}.$$

This system as $SO(2N)$ symmetry, and accordingly has $N(2N-1)$ currents $J^{[ij]}(z) = \psi^i(z)\psi^j(z)$, where the brackets denote anti-symmetry.

(a) Compute the OPE of the currents $J^{[ij]}$ and show that the form a closed OPE algebra (i.e. the singular terms in the OPE can be expressed in terms of the currents and the identity operator). This is known as the $SO(2N)$ Kac-Moody algebra.

(b) If we combine ψ^{2i-1} and ψ^{2i} into a complex fermion ψ^a, $a = 1, \dots, N$, we can bosoniza the system to a collection of N left-moving scalars ϕ^a with an OPE

$$\phi^a(z)\phi^b(w) \sim -\log(z-w).$$

Write down the expressions for the $SO(2N)$ currents in terms of scalar fields ϕ^a and verify that they satisfy the same OPE algebra as $J^{[ij]}$.