1. Consider an anti-symmetric tensor field with $p$ indices, $A_{i_1 \ldots i_p}$, or equivalently a $p$-form $A = \frac{1}{p!} A_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p}$, with an action

$$S = \int_{\mathbb{R}^D} dA \wedge * dA,$$

where $*$ is the Hodge star. This theory has a $(p-1)$-form gauge invariance $A \mapsto A + d\lambda$, where $\lambda$ is an arbitrary $(p-1)$ form field. For $p = 1$ this theory describes a massless vector particle (photon) in $D$ dimensions, and it is well-known that a photon has $(D - 2)$ polarization states.

(a) Define an analog of the Lorenz gauge for general $p$ and write the mode expansion for the field $A$ in this gauge. Show that for a fixed light-like $D$-momentum $k$ an analog of the polarization vector is a $p$-form $\zeta$ satisfying $k_i \zeta^{i_1 \ldots i_p} = 0$. Show that there is a residual gauge symmetry and determine how $\zeta$ transforms under this symmetry.

(b) Show that physical polarization states transform as a rank-$p$ anti-symmetric tensor of the "little group" $SO(D - 2)$. In particular, show that for $D = 10$, the particle corresponding to a 2-form field has 28 polarizations states, and the particle corresponding to a 3-form field has 35 polarization states.

2. Consider $2N$ free left-moving real fermions $\psi^i$, $i = 1, \ldots, 2N$, with the standard OPE

$$\psi^i(z) \psi^j(w) \sim \frac{\delta^{ij}}{z - w}.$$

This system as $SO(2N)$ symmetry, and accordingly has $N(2N - 1)$ currents $J^{ij}(z) = \psi^i(z) \psi^j(z)$, where the brackets denote anti-symmetry.

(a) Compute the OPE of the currents $J^{ij}$ and show that the form a closed OPE algebra (i.e. the singular terms in the OPE can be expressed in terms of the currents and the identity operator). This is known as the $SO(2N)$ Kac-Moody algebra.

(b) If we combine $\psi^{2i-1}$ and $\psi^{2i}$ into a complex fermion $\psi^a$, $a = 1, \ldots, N$, we can bosoniza the system to a collection of $N$ left-moving scalars $\phi^a$ with
an OPE

\[ \phi^a(z) \phi^b(w) \sim -\log(z - w). \]

Write down the expressions for the \( SO(2N) \) currents in terms of scalar fields \( \phi^a \) and verify that they satisfy the same OPE algebra as \( J^{[ij]} \).