Solutions to HW1

1.2. The equation of motion (1.2.23) implies $T^{00} = 0$. From eq. (1.2.22) this is the same as $\partial_0 X^\mu \partial_0 X_\mu + \partial_1 X^\mu \partial_1 X_\mu = 0$. At the endpoint of a string, $\partial_1 X^\mu = 0$, thus the equation of motion implies $\partial_0 X^\mu \partial_0 X_\mu = 0$. That is, the velocity vector $\partial_0 X^\mu$ is light-like.

1.4. The mass $m^2 = 1/\alpha'$ corresponds to the operator N (see eq. (1.3.37)) having eigenvalue 2. There are two ways to get $N = 2$: (1) $N_1 = 2$, all other N_n vanish; (2) $N_2 = 1$, all other N_n vanish. The two kind of states are

$$C^{ij} = \alpha^i_{-1}\alpha^j_{-1}|0\rangle, \quad C^i = \alpha^i_{-2}|0\rangle.$$

The indices here are "transverse", i.e. run from 2 to $D-1$. C^{ij} is a symmetric tensor, which can be decomposed into a symmetric and traceless $C^{(ij)}$ and a scalar C.

On the other hand, we can consider massive representations of the Poincare group in D dimensions. Going to the rest frame, we see that the "little group" is $SO(D-1)$ which rotates the coordinates $x^1, \ldots, x^{D-1}$. Let us denote the indices of the vector representation of $SO(D-1)$ by upper case letters $I, J, K, ...$ If the above spectrum of states is to be obtained from a representation of $SO(D-1)$, it must contain a traceless symmetric tensor $B^{(IJ)}$. It decomposes under $SO(D-2)$ into a traceless symmetric B^{ij}, a scalar B, and a vector B^{i0}. They can be identified with $C^{(ij)}$, C and C^i above. So the open-string states at level $N = 2$ assemble into a traceless symmetric rank-2 tensor of $SO(D-1)$.

The mass $m^2 = 2/\alpha'$ corresponds to $N = 3$. There are three kind of states at this level:

$$D^{ijk} = \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}|0\rangle, \quad D^{ij} = \alpha^i_{-2}\alpha^j_{-1}|0\rangle, \quad D^i = \alpha^i_{-3}|0\rangle.$$

The tensor D^{ijk} is symmetric, the tensor D^{ij} is not.

Clearly, to reproduce this spectrum, we need to have a traceless symmetric rank-3 tensor $E^{(IJK)}$ which decomposes into E^{ijk} (symmetric but not traceless) and E^{ij} (symmetric but not traceless). This accounts for D^{ijk} and the symmetric part of D^{ij}. The anti-symmetric part of D^{ij} and D^i form an irreducible representation of $SO(D-1)$, namely an anti-symmetric tensor $E^{[IJ]}$. So the open-string states at level $N = 3$ assemble into a traceless symmetric rank-3 tensor and an anti-symmetric rank-2 tensor of $SO(D-1)$. They have exactly the same mass.