Ph250a: Solutions to Homework 2

Problem 1.

Action for the Maxwell field in any dimension is

$$S = \int d^D x \sqrt{-\det g}\, \mathcal{L} = -\frac{1}{4} \int d^D x \sqrt{-\det g}\, g^{\mu\nu} g^{\nu\rho} F_{\mu\nu} F_{\nu\rho}. \tag{1}$$

Variation of this action with respect to metric is given by

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-\det g}} \frac{\delta\left(\sqrt{-\det g}\mathcal{L}\right)}{\delta g^{\mu\nu}} = -F_{\mu\sigma} F^{\sigma}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} \tag{2}$$

From this we find the trace of energy-momentum tensor

$$T^{\mu}_{\mu} = \frac{1}{4}(D-4) F_{\mu\nu} F^{\mu\nu} \tag{3}$$

which is equal to $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ for $D = 3$. By varying the action with respect to A_μ one finds the usual Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0 \tag{4}$$

$$\epsilon_{\mu\nu\rho} \partial_\mu F_{\nu\rho} = 0 \tag{5}$$

which have solutions such that $F_{\mu\nu} F^{\mu\nu} \neq 0$ and therefore the trace of the energy momentum tensor is non-zero even on-shell.

We can try to modify the energy momentum tensor by adding derivative of a 3-tensor $B_{\mu\nu\rho}$ which is antisymmetric in first two indices

$$T_{\mu\nu} \to T_{\mu\nu} + \partial_\rho B^{\rho}{}_{\mu\nu} \tag{6}$$

But since $F_{\mu\nu} F^{\mu\nu}$ is not a total derivative of any gauge invariant quantity it is impossible to make energy-momentum tensor traceless.

Problem 2.

Under small dilation the field change as (I use the conventions where field transformation due to coordinate change is subtracted from the field transformation law see sec. 2.4.2 in Di Francesco et. al CFT)

$$x^\mu \to (1+\alpha)x^\mu \tag{7}$$

$$A^\mu \to (1-\alpha\Delta)A^\mu \tag{8}$$

where Δ is the dimension of the field. Using the Noether theorem

$$j^\mu_{(D)} = -\mathcal{L}x^\mu + \frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}x^\rho\partial_\rho A_\nu + \frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\rho)}\Delta A_\rho = T^\mu_\rho x^\rho - F_{\mu\rho}\partial_\rho A^\nu x_\nu - F^{\mu\rho}\Delta A_\rho \tag{9}$$

It's conservation law is

$$\partial_\mu j^\mu_{(D)} = \partial_\mu T^{\mu\nu}x_\nu + T^\mu_\mu - \partial_\mu F_{\mu\rho}\partial_\rho A^\nu x_\nu - \Delta\,\partial_\mu F^{\mu\nu}A_\nu + \frac{1}{2}\left(1-\Delta\right)F^{\mu\nu}F_{\mu\nu} - F_{\mu\rho}\partial_\rho A^\nu x_\nu \tag{10}$$

Using equations of motion, results of the previous problem and energy-momentum conservation we get in $D=3$

$$\partial_\mu j^\mu_{(D)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(1-\Delta\right)F^{\mu\nu}F_{\mu\nu} \tag{11}$$

So the dilatation current will be conserved if we choose $\Delta = \frac{1}{2}$