Ph250a: Solutions to Homework 3

Problem 1.

The field $\phi(z)$ has Laurent series expansion

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h} \tag{1}$$

where modes ϕ_n satisfy $\phi_n|0\rangle = 0$ for $n > -h$. Therefore,

$$\phi(0)|0\rangle = \phi_{-h}|0\rangle \equiv |\phi\rangle \tag{2}$$

where singular terms in the Laurent series are zero due to condition $\phi_n|0\rangle = 0$ for $n > -h$ and non-singular are zero at $z = 0$ leaving only ϕ_{-h} contribution. Action of L_n on the state $|\phi\rangle$ by definition is given by

$$L_n|\phi\rangle \equiv \int_{|z|=\delta} \frac{dz}{2\pi i} z^{n+1} T(z)|\phi\rangle \tag{3}$$

which is equal to

$$\int_{|z|=\delta} \frac{dz}{2\pi i} z^{n+1} T(z)\phi(0)|0\rangle \tag{4}$$

by state-operator correspondence. Since ϕ is a primary field it has the following OPE with $T(z)$

$$T(z)\phi(0) \sim \frac{h\phi(0)}{z^2} + \frac{\partial\phi(0)}{z} + \dots \tag{5}$$

Substituting this expansion in (4) we have

$$\int_{|z|=\delta} \frac{dz}{2\pi i} z^{n+1} T(z)\phi(0)|0\rangle = \int_{|z|=\delta} \frac{dz}{2\pi i} z^{n+1} \left(\frac{h\phi}{z^2} + \frac{\partial\phi}{z} + \dots \right) |0\rangle \tag{6}$$

Since the integrand is regular at $z = 0$ for $n > 0$ the integral vanishes.

Problem 2.

Let us assume that OPE $T(z)$ and $\phi(0)$ has the following form

$$T(z)\phi(0) = \sum_{n\in\mathbb{Z}} \frac{A_n}{z^{n+2}} \tag{7}$$

where A_n are operator valued coefficients of Laurent series. Using the same manipulation as in previous problem we get

$$L_n|\phi\rangle = \int_{|z|=\delta} \frac{dz}{2\pi i} z^{n+1} T(z)\phi(0)|0\rangle = \sum_{m\in\mathbb{Z}} \int_{|z|=\delta} \frac{dz}{2\pi i} z^{n-m-1} A_m|0\rangle \tag{8}$$

This integral is non-zero only when integrand is proportional to $\frac{1}{z}$.

$$L_n|\phi\rangle = \sum_{m\in\mathbb{Z}} \int_{|z|=\delta} \frac{dz}{2\pi i} z^{n-m-1} A_m|0\rangle = \sum_{m\in\mathbb{Z}} \delta_{n,m} A_m|0\rangle = A_n|0\rangle = |A_n\rangle \tag{9}$$

where in the last step we used state-operator correspondence. From $L_n|\phi\rangle = 0$ for $n > 0$ we see that all terms with poles of order greater than 2 vanish in (7). From $L_0|\phi\rangle = h|\phi\rangle$ we see that $A_0 = h\phi(0)$. And from $L_{-1}|\phi\rangle = L_{-1}\phi(0)|0\rangle = [L_{-1}, \phi(0)]|0\rangle = \partial\phi(0)|0\rangle$ we find that $A_{-1} = \partial\phi(0)$. Therefore the OPE looks like

$$T(z)\phi(0) \sim \frac{h\phi(0)}{z^2} + \frac{\partial\phi(0)}{z} + \dots \tag{10}$$

which implies that ϕ is a primary field of dimension h.