Problem 1.

Let us first consider state corresponding to \( c(z) \) and find what modes of \( b \) annihilate it. From the contour integral

\[
\langle b_n | c \rangle = \int \frac{dz}{2\pi i z} z^{n+2} b(z) c(0) \langle 0 | = \int \frac{dz}{2\pi i z} z^{n+2} \left( \frac{1}{z} + \ldots \right) \langle 0 |
\]

we see that \( \langle b_n | c \rangle = 0 \) iff \( n \geq 0 \), i.e. it is the same as the state \( | \downarrow \rangle \). The same conclusion follows if we apply the usual state-operator correspondence \((\phi, |0\rangle = |\phi\rangle)\) to the vacuum \( |0\rangle = b_{-1} | \downarrow \rangle \).

For the state corresponding to \( b(z) \) we find

\[
\langle c_n | b \rangle = \int \frac{dz}{2\pi i z} z^{n-1} c(z) b(0) \langle 0 | = \int \frac{dz}{2\pi i z} z^{n-1} \left( \frac{1}{z} + \ldots \right) \langle 0 |
\]

i.e. \( \langle c_n | b \rangle = 0 \) iff \( n \geq 3 \). It is the same state as \( b_{-2} b_{-1} | \downarrow \rangle \).

We have already determined the operator corresponding to \( | \downarrow \rangle \). In order to find operator corresponding to \( | \uparrow \rangle \) we notice that it should not be annihilated by both \( b_1 \) and \( b_2 \) modes (this means it must have first and second order pole in OPE with \( b \) and also it must have ghost charge 2 greater than the vacuum operator. Therefore, a natural guess is : \( c \partial c : (z) \). Direct check confirms this.

Problem 2.

The general state on the second level looks like

\[
|\phi\rangle = (A_\mu \alpha^\mu_1 \alpha^\nu_{-1} + B_\mu \alpha^\mu_{-2}) |0, p\rangle
\]
Before checking the physical condition on this state let us write the relevant terms in the Virasoro operators

\[ L_0 = \alpha' p^2 + \alpha_{-1} \cdot \alpha_{-1} + \alpha_{-2} \cdot \alpha_{-2} + \ldots \]  
(4)

\[ L_1 = \sqrt{2\alpha' p} \cdot \alpha_1 + \alpha_{-1} \cdot \alpha_2 + \ldots \]  
(5)

\[ L_2 = \sqrt{2\alpha' p} \cdot \alpha_2 + \frac{1}{2} \alpha_1 \cdot \alpha_1 + \ldots \]  
(6)

\[ L_{-1} = \sqrt{2\alpha' p} \cdot \alpha_{-1} + \alpha_{-2} \cdot \alpha_1 + \ldots \]  
(7)

\[ L_{-2} = \sqrt{2\alpha' p} \cdot \alpha_{-2} + \frac{1}{2} \alpha_{-1} \cdot \alpha_{-1} + \ldots \]  
(8)

Let us now write down Virasoro constraints. The first one reads

\[ (L_0 - a) |\phi\rangle = (\alpha' p^2 + 2 - a) |\phi\rangle \]  
(9)

which can be solved by

\[ \alpha' m^2 \equiv -\alpha' p^2 = 2 - a. \]  

The second constraint reads

\[ L_1 |\phi\rangle = 2(\sqrt{2\alpha' A_{\mu\nu} p^\mu + B_\nu}) \alpha^-_{-1} |0, p\rangle \]  
(10)

which can be solved by

\[ B_\mu = -\sqrt{2\alpha'} A_{\mu\nu} p^\nu \]  
(11)

The last constraint is

\[ L_2 |\phi\rangle = (A_\mu^\mu + 2\sqrt{2\alpha'} p \cdot B) \]  
(12)

i.e. tensor \( A_{\mu\nu} \) must satisfy

\[ A_\mu^\mu = 4\alpha' p \cdot A \cdot p \]  
(13)

The general spurious state on this level looks like

\[ |\chi\rangle = (L_{-1} \beta_\mu \alpha_{-1}^\mu + \gamma L_{-2}) |0, p\rangle = \left( \sqrt{2\alpha'} \beta_\mu p_\nu + \frac{\gamma}{2} \eta_{\mu\nu} \right) \alpha_{-1}^\mu \alpha_{-1}^\nu |0, p\rangle + \left( \beta_\mu + \sqrt{2\alpha'} \gamma p_\mu \right) \alpha_{-2}^\mu |0, p\rangle \]  
(14)

Not all of these states are physical. The condition for the spurious state to become physical is easier to do in the rest-frame where \( p_\mu = (m, 0, 0, \ldots, 0) \). The conditions are

\[ \left( -4\alpha' m^2 + \frac{D}{2} \right) \gamma = 3\sqrt{2\alpha'} \beta_0 m \]  

\[ (1 - \alpha' m^2) \beta_i = 0 \]  

\[ 2(1 - \alpha' m^2) \beta_0 = \left( \sqrt{2\alpha' m} \beta_0 - 3\gamma \right) \sqrt{2\alpha'} \]  

where \( \beta_0 \) and \( \beta_i \) are time and space parts of \( \beta_\mu \).
Now we can analyse the spectrum. For general $D$ and $a$ there are no spurious states and the number of physical states is $\frac{D(D+1)}{2} - 1$. This can be decomposed as $\frac{D(D+1)}{2} - 1 = (\frac{D(D-1)}{2} - 1) + (D - 1) + (1)$, i.e. traceless symmetric + vector + scalar representations of $SO(D-1)$. From the equations (15) one can see that when $a = 1$ the vector spurious state become null removing the vector particle. And also for all values of $D$ there exist $a$ such that scalar particle become null ($a = 1$ and $D = 26$ is an example of this).

Let us write down these states explicitly

\begin{align*}
\text{traceless symmetric: } & \quad A_{\mu\nu} p^\nu = 0 \quad A_\mu^\mu = 0 \quad B_\mu = 0 \\
\text{vector: } & \quad A_{\mu\nu} = p_\mu v_\nu + p_\nu v_\mu \quad B_\mu = \sqrt{2\alpha'^2 m^2} v_\mu \quad p \cdot v = 0 \\
\text{scalar: } & \quad A_\nu = \frac{1 + 4\alpha'^2 m^2}{D + 4\alpha'^2 m^2} \delta_\nu^\mu + \frac{p_\mu p_\nu}{m^2} \quad B_\mu = \frac{\sqrt{2\alpha'} (D - 1)}{D + 4\alpha'^2 m^2} p_\mu
\end{align*}

The norm can be found to be

$$2A_{\mu\nu} A_{\mu\nu} + 2B_\mu B^\mu$$

Since traceless symmetric tensor has no time-like components in the rest frame its norm is always positive. The norm of the vector particle is given by

$$2A_{\mu\nu} A_{\mu\nu} + 2B_\mu B^\mu = 4v^2 m^2 (\alpha'^2 m^2 - 1) = \frac{4v^2}{\alpha'} (2 - a)(1 - a)$$

Therefore the norm of this state is positive for $a < 1$ or $a > 2$ and negative in the interval $1 < a < 2$.

The norm of the scalar field is given by

$$2A_{\mu\nu} A_{\mu\nu} + 2B_\mu B^\mu = \frac{(D - 1)(2a - 3)(D - 1) + (9 - 4a)^2}{(D + 8 - 4a)^2}$$

For the special case $a = 1$ it becomes

$$\frac{(D - 1)(26 - D)}{(D + 4)^2}$$

So the norm is negative for $D > 26$. At $D = 26$ this state has zero norm and removed from the spectrum by the ghosts.