Ph250a: Solutions to Homework 5

Problem 1.

Let us first consider state corresponding to $c(z)$ and find what modes of b annihilate it. From the contour integral

$$b_n|c\rangle = \int \frac{dz}{2\pi i z} z^{n+2} b(z)c(0)|0\rangle = \int \frac{dz}{2\pi i z} z^{n+2} \left(\frac{1}{z} + \dots\right)|0\rangle \tag{1}$$

we see that $b_n|c\rangle = 0$ iff $n \geq 0$, i.e. it is the same as the state $|\downarrow\rangle$. The same conclusion follows if we apply the usual state-operator correspondence ($\phi_{-h}|0\rangle = |\phi\rangle$) to the vacuum $|0\rangle = b_{-1}|\downarrow\rangle$.

For the state corresponding to $b(z)$ we find

$$c_n|b\rangle = \int \frac{dz}{2\pi i z} z^{n-1} c(z)b(0)|0\rangle = \int \frac{dz}{2\pi i z} z^{n-1} \left(\frac{1}{z} + \dots\right)|0\rangle \tag{2}$$

i.e. $c_n|b\rangle = 0$ iff $n \geq 3$. It is the same state as $b_{-2}b_{-1}|\downarrow\rangle$.

We have already determined the operator corresponding to $|\downarrow\rangle$. In order to find operator corresponding to $|\uparrow\rangle$ we notice that it should not be annihilated by both b_1 and b_2 modes (this means it must have first and second order pole in OPE with b) and also it must have ghost charge 2 greater than the vacuum operator. Therefore, a natural guess is $:c\partial c:(z)$. Direct check confirms this.

Problem 2.

The general state on the second level looks like

$$|\phi\rangle = \left(A_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + B_{\mu}\alpha^{\mu}_{-2}\right)|0,p\rangle \tag{3}$$

Before checking the physical condition on this state let us write the relevant terms in the Virasoro operators

$$L_0 = \alpha' p^2 + \alpha_{-1} \cdot \alpha_{-1} + \alpha_{-2} \cdot \alpha_{-2} + \dots \tag{4}$$
$$L_1 = \sqrt{2\alpha'} p \cdot \alpha_1 + \alpha_{-1} \cdot \alpha_2 + \dots \tag{5}$$
$$L_2 = \sqrt{2\alpha'} p \cdot \alpha_2 + \frac{1}{2} \alpha_1 \cdot \alpha_1 + \dots \tag{6}$$
$$L_{-1} = \sqrt{2\alpha'} p \cdot \alpha_{-1} + \alpha_{-2} \cdot \alpha_1 + \dots \tag{7}$$
$$L_{-2} = \sqrt{2\alpha'} p \cdot \alpha_{-2} + \frac{1}{2} \alpha_{-1} \cdot \alpha_{-1} + \dots \tag{8}$$

Let us now write down Virasoro constraints. The first one reads

$$(L_0 - a)|\phi\rangle = (\alpha' p^2 + 2 - a)|\phi\rangle \tag{9}$$

which can be solved by $\alpha' m^2 \equiv -\alpha' p^2 = 2 - a$. The second constraint reads

$$L_1|\phi\rangle = 2(\sqrt{2\alpha'} A_{\mu\nu} p^\mu + B_\nu) \alpha^\nu_{-1} |0, p\rangle \tag{10}$$

which can be solved by

$$B_\mu = -\sqrt{2\alpha'} A_{\mu\nu} p^\nu \tag{11}$$

The last constraint is

$$L_2|\phi\rangle = (A^\mu_\mu + 2\sqrt{2\alpha'} p \cdot B) \tag{12}$$

i.e. tensor $A_{\mu\nu}$ must satisfy

$$A^\mu_\mu = 4\alpha' p \cdot A \cdot p \tag{13}$$

The general spurious state on this level looks like

$$|\chi\rangle = \left(L_{-1} \beta_\mu \alpha^\mu_{-1} + \gamma L_{-2}\right) |0, p\rangle = \left(\sqrt{2\alpha'} \beta_\mu p_\nu + \frac{\gamma}{2} \eta_{\mu\nu}\right) \alpha^\mu_{-1} \alpha^\nu_{-1} |0, p\rangle + \left(\beta_\mu + \sqrt{2\alpha'} \gamma p_\mu\right) \alpha^\mu_{-2} |0, p\rangle \tag{14}$$

Not all of these states are physical. The condition for the spurious state to become physical is easier to do in the rest-frame where $p_\mu = (m, 0, 0, \dots, 0)$. The conditions are

$$\begin{aligned} \left(-4\alpha' m^2 + \frac{D}{2}\right) \gamma &= 3\sqrt{2\alpha'} \beta_0 m \\ (1 - \alpha' m^2) \beta_i &= 0 \\ 2(1 - \alpha' m^2) \beta_0 &= \left(\sqrt{2\alpha'} m \beta_0 - 3\gamma\right) \sqrt{2\alpha'} m \end{aligned} \tag{15}$$

where β_0 and β_i are time and space parts of β_μ.

Now we can analyse the spectrum. For general D and a there are no spurious states and the number of physical states is $\frac{D(D+1)}{2} - 1$. This can be decomposed as $\frac{D(D+1)}{2} - 1 = (\frac{D(D-1)}{2} - 1) + (D-1) + (1)$, i.e. traceless symmetric + vector + scalar representations of $SO(D-1)$. From the equations (15) one can see that when $a = 1$ the vector spurious state become null removing the vector particle. And also for all values of D there exist a such that scalar particle become null ($a = 1$ and $D = 26$ is an example of this).

Let us write down these states explicitly

$$\text{traceless symmetric:} \quad A_{\mu\nu}p^{\nu} = 0 \quad A^{\mu}_{\mu} = 0 \quad B_{\mu} = 0 \tag{16}$$

$$\text{vector:} \quad A_{\mu\nu} = p_{\mu}v_{\nu} + p_{\nu}v_{\mu} \quad B_{\mu} = \sqrt{2\alpha'}m^2 v_{\mu} \quad p \cdot v = 0 \tag{17}$$

$$\text{scalar:} \quad A^{\mu}_{\nu} = \frac{1 + 4\alpha' m^2}{D + 4\alpha' m^2}\delta^{\mu}_{\nu} + \frac{p^{\mu}p_{\nu}}{m^2} \quad B_{\mu} = \frac{\sqrt{2\alpha'}(D-1)}{D + 4\alpha' m^2}p_{\mu} \tag{18}$$

The norm can be found to be

$$2A^{\mu\nu}A_{\mu\nu} + 2B_{\mu}B^{\mu} \tag{19}$$

Since traceless symmetric tensor has no time-like components in the rest frame its norm is always positive. The norm of the vector particle is given by

$$2A^{\mu\nu}A_{\mu\nu} + 2B_{\mu}B^{\mu} = 4v^2 m^2(\alpha' m^2 - 1) = 4\frac{v^2}{\alpha'}(2-a)(1-a) \tag{20}$$

Therefore the norm of this state is positive for $a < 1$ or $a > 2$ and negative in the interval $1 < a < 2$.

The norm of the scalar field is given by

$$2A^{\mu\nu}A_{\mu\nu} + 2B_{\mu}B^{\mu} = \frac{(D-1)[(2a-3)(D-1) + (9-4a)^2]}{(D+8-4a)^2} \tag{21}$$

For the special case $a = 1$ it becomes

$$\frac{(D-1)(26-D)}{(D+4)^2} \tag{22}$$

So the norm is negative for $D > 26$. At $D = 26$ this state has zero norm and removed from the spectrum by the ghosts.