Homework 2

1. (30 pts) (a) A supersymmetric quantum mechanics is a QM system described in the path-integral formalism using maps from a super-worldline with coordinates (t, θ) to the target X (a Riemannian manifold with metric $g_{\mu\nu}$). The coordinate t is bosonic, while θ is fermionic, $\theta^2 = 0$. Thus the only observable in the theory is a superfield $\Phi^\mu(t, \theta)$, $i = 1, \ldots, \dim X$.The action looks as follows:

$$S = \int dt d\theta \; \partial_t \Phi^\mu D_\theta \Phi^\nu g_{\mu\nu}(\Phi).$$

Here $D_\theta = \partial_\theta + i\theta\partial_t$.

Taylor-expand the superfield $\Phi(t, \theta)$ in θ:

$$\Phi^\mu(t, \theta) = \phi^\mu(t) + \theta\psi^\mu(t)$$

and express the action in terms of ordinary fields ϕ^μ and ψ^μ.

(b) The supersymmetry transformation in terms of the superfield Φ^μ looks as follows:

$$\delta_Q \Phi^\mu = (\partial_\theta - i\theta\partial_t)\,\Phi^\mu.$$

Write down the transformation in terms of the component fields ϕ^μ and ψ^μ. Write down the Noether charge corresponding to this symmetry.

(c) Quantize the theory in part (a) and show that the Hilbert space can be identified with the space of sections of a (chiral) spinor bundle over X, while the supersymmetry operator can be identified with the Dirac operator.

2. (10 pts) Consider $N = (1, 1)$ supersymmetric sigma-model on a world-sheet with a boundary $\sigma^1 = 0$ and target space X. Let the boundary condition for the bosonic field $X^\mu(\sigma^0, \sigma^1)$ be

$$g_{\mu\nu}(X)\partial_0 X^\nu = F_{\mu\nu}(X)\partial_1 X^\nu.$$

Here $g_{\mu\nu}(X)$ is a metric on X and $F_{\mu\nu}(X) = \partial_\mu A_\nu(X) - \partial_\nu A_\mu(X)$, and $A_\mu(X)$ is a vector potential on the target space X. Determine the boundary conditions on the fermions which preserve some supersymmetry. Assume that the boundary condition is linear in fermions: $\psi_+^\mu = R^\mu_\nu \psi_-^\nu$ for some matrix R (which may be a function on X). Note that it will be impossible to preserve both left-moving and right-moving supersymmetry, one can preserve only their sum. Hint: write down the supercurrents for left-handed and right-handed supersymmetries and require their spatial components to differ by a sign on the boundary. This will ensure the conservation of the sum of the supercharges.