Homework 3

1. (a) (20 pts) One can discretize the free massless boson field in 2d as follows. Consider a square lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ and place a real variable $\phi(v)$ on each vertex $v \in \mathbb{Z}^2$. Consider an action

$$S(\phi) = \frac{1}{2}\sum_e (\phi(v_+(e)) - \phi(v_-(e))^2,$$

where the sum is over all edges e of the lattice and $v_+(e)$ and $v_-(e)$ are the two vertices of the edge e. Let the partition function be

$$Z(\beta) = \sum_\phi e^{-\beta S(\phi)}.$$

This is ill-defined, for two reasons. First, there is a divergence arising from an infinite number of vertices. To avoid this, let us work on a torus of size $L \times L$ instead of $\mathbb{R}^2$. Thus we only have L^2 vertices. Second, the integrand is invariant under a shift $\phi(v) \mapsto \phi(v) + \epsilon$, so there is a divergence arising from the non-compactness of the range of the variables $\phi(v)$. We will just ignore this divergence.

By mimicking the continuum argument for T-duality, show that $Z(\beta)$ and $Z(1/\beta)$ are proportional and determine how the proportionality coefficient depends on β and L. Be careful: on a torus not every closed 1-form is exact, and a similar statement is true for our discretized torus.

(b) (10 pts) Consider the observable $W_\alpha(v) = e^{i\alpha\phi(v)}$, where α is a real number. This is a lattice analog of the vertex operator $e^{ipX(z,\bar{z})}$. Repeat the manipulations of part (a) for the expectation value

$$\langle W_\alpha(v) W_{-\alpha}(v') \rangle$$

and determine the image of $W_\alpha(v)$ under T-duality.