Fermionic phases of matter and spin structures

Anton Kapustin

California Institute of Technology

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Outline

- Gapped phases of matter and TQFT
- Fermions vs. bosons
- Fermionic phases in 1+1D
- Fermionic phases in 2+1D

References: Z-C. Gu and X-G. Wen (1201.2648), D. Gaiotto and A.K. (1505.05856). Also older work by Gu-Wang-Wen, Fidkowski-Kitaev, Chen-Gu-Wen.

Gapped phases of matter

In grade school we learn about the following phases: solid, liquid, gas.

Actually, liquid $\simeq$ gas, and there are many crystalline solids distinguished by their symmetries.

At low temperatures, the basic feature is the presence/absence of a gap between the ground state and the 1st excited state.

If the gap is nonzero even for an infinite system, the phase is called gapped.

Examples of gapless phases: crystals, superfluid ^4He, Fermi liquid.

Examples of gapped phases: broken discrete symmetry, Quantum Hall phases, confining and Higgs phases of gauge theories.

Topological Phases of Matter

Old viewpoint: gapped phases of matter are “boring” because they all look the same at long distances/time-scales.

Modern viewpoint: there is a variety of topological phases of matter which are gapped.

To distinguish them one can either consider a nontrivial spatial topology or to look at the edge physics.

Example: Fractional Quantum Hall states. (Ground-state degeneracy on a space of nontrivial topology, gapless edge modes). IR physics is described by a nontrivial 3D Topological Quantum Field Theory (TQFT).

Classifying gapped phases

Want to classify "gapped" "local" "lattice" Hamiltonians up to homotopy.

"Lattice": the Hilbert space is a tensor product

$$\mathcal{H} = \otimes_{v\in V}\mathcal{H}_v$$

where V is the set of vertices of a d-dimensional lattice or triangulation, and $\mathcal{H}_v$ is a finite-dimensional Hilbert space.

"Local": the Hamiltonian has the form

$$H = \sum_v H_v,$$

where H_v acts as identity on all $\mathcal{H}_{v'}$, except for v' in a neighborhood of v.

"Gapped": the gap between the energies of ground state(s) and excited states stays nonzero in the limit of infinite volume.

Gapped phases and TQFTs

It is believed that gapped phases can be classified using TQFT.

Namely, the IR limit of a gapped phase is described by a TQFT, and it carries complete information about the phase.

Hard to check for $d \geq 2$ because neither TQFTs nor gapped phases have been classified.

One can simplify the problem by imposing invertibility, or complicate it by adding symmetry.

Reminder on TQFTs

A TQFT in $d+1$ dimensions attaches

- to a closed oriented $(d+1)$-manifold M, a complex number $Z(M)$ (the partition function)
- to a closed oriented d-manifold N, a f.d. vector space $\mathcal{H}(N)$
- to a closed oriented $(d-1)$-manifold K, a linear category $\mathcal{C}(K)$
- ...

There are also rules for assigning algebraic objects to manifolds with boundaries and corners.

Axioms:

diffeomorphism-invariance, disjoint union maps to product, gluing

Invertible Gapped Phases and TQFTs

One can simplify the problem by restricting to invertible phases and invertible TQFTs.

Both gapped phases and TQFTs form a monoid (a set with an associative binary operation and a neutral element).

The operation corresponds to "stacking" systems together, the neutral element is the trivial phase (TQFT).

Not all phases (TQFTs) have an inverse; those which do are called invertible.

An invertible phase (TQFT) has a unique state (i.e. one dimensional space of states) for any closed spatial geometry.

Symmetry-Enhanced Phases and TQFTs

One can refine the classification problem by requiring the phase to have a symmetry G. I will assume that G is finite, for simplicity.

A lattice model has a symmetry G if each $\mathcal{H}_v$ is a representation of G, and the action of G on $\otimes_v \mathcal{H}_v$ commutes with the Hamiltonian.

To define TQFTs with a symmetry G, we replace closed oriented manifolds with closed oriented manifolds equipped with G-bundles. Or equivalently, with closed oriented manifolds equipped with maps to $BG = K(G, 1)$.

Thus, a G-equivariant TQFT attaches a vector space $\mathcal{H}(N, P)$ to every G-bundle P over a closed oriented d-manifold N. If the TQFT is invertible, this vector space is one-dimensional.

Bosoni-Fermion dichotomy

An important modification: allow $\mathcal{H}_v$ to be $\mathbb{Z}_2$-graded. Physically, the grading operator is $(-1)^F$.

Vectors in $\mathcal{H}_v$ with $(-1)^F = -1$ are called fermionic, vectors with $(-1)^F = 1$ are called bosonic.

The tensor product $\otimes_v \mathcal{H}_v$ is also understood in the $\mathbb{Z}_2$-graded sense (i.e. fermionic operators localized at different v anti-commute). The Hamiltonian is still even.

The symmetry group G commutes with $(-1)^F$.

What is the analogous modification on the TQFTs side?

Spin TQFTs

In a unitary relativistic QFT, statistics correlates with spin. Could spin-TQFTs classify gapped fermionic phases?

Spin-TQFT attaches numbers, vector spaces, categories, . . . , to closed manifolds with spin structure.

Spin-TQFTs have not been much studied, even in low dimensions. In what follows I will explain how to construct spin-TQFTs in 2D and (if time permits) in 3D.

What is a spin structure?

A spin structure on an oriented manifold M is similar, but not the same as a $\mathbb{Z}/2$ gauge field on M.

To explain this, let me remind some mathematical terminology.

Bluff your way into (co)chains and (co)cycles

We will think about (co)homology groups in terms of triangulations. A k-chain E with values in an abelian group $\mathfrak{A}$ is an $\mathfrak{A}$-valued function on k-simplices. The group of k-chains is denoted $C_k(M, \mathfrak{A})$. The group of cochains $C^k(M, \mathfrak{A})$ is the same as $C_k(M, \mathfrak{A})$.

In this talk, we will make use of $\mathfrak{A} = \mathbb{Z}/2$ and $\mathfrak{A} = \mathbb{R}/\mathbb{Z} \simeq U(1)$.

There is a boundary operator $\partial_k : C_k(M, \mathfrak{A}) \to C_{k-1}(M, \mathfrak{A})$ and a coboundary operator $\delta_k : C^k(M, \mathfrak{A}) \to C^{k+1}(M, \mathfrak{A})$. They satisfy $\partial^2 = 0$, $\delta^2 = 0$. A chain annihilated by ∂ is called a cycle, a cochain annihilated by δ is called a cocycle. The subgroup of k-cycles is denoted $Z_k(M, \mathfrak{A})$, the subgroup of k-cocycles is denoted $Z^k(M, \mathfrak{A})$.

One defines $H_k(M, \mathfrak{A}) = \mathrm{ker}\partial_k/\mathrm{im}\partial_{k+1}$, $H^k(M, \mathfrak{A}) = \mathrm{ker}\delta_k/\mathrm{im}\delta_{k-1}$.

Bluff your way into spin structures I

A spin structure on an oriented D-manifold is a way to parallel-transport spinors. It enables one to define the Dirac operator.

More formally, it is way to lift a principal $SO(D)$-bundle of oriented orthonormal frames to a principal $Spin(D)$-bundle.

For $D > 3$, not every oriented D-manifold admits a spin structure: it exists iff $w_2(M) = 0$, where $w_2 \in H^2(M, \mathbb{Z}/2)$ is the 2nd Stiefel-Whitney class of the tangent bundle of M. For $D \leq 3$ the class w_2 vanishes automatically.

Any two spin structures differ by a $\mathbb{Z}/2$ gauge field, i.e. an element of $H^1(M, \mathbb{Z}/2)$. Thus the set of spin structures can be identified with $H^1(M, \mathbb{Z}/2)$, but not canonically.

Bluff your way into spin structures II

- On a circle, there are two spin structures because $H^1(S^1, \mathbb{Z}/2) = \mathbb{Z}/2$. In this case there is a canonical identification with $\mathbb{Z}/2$ gauge fields on S^1. The non-trivial $\mathbb{Z}/2$ gauge field corresponds to the Neveu-Schwarz (NS) spin structure, the trivial one to the Ramond (R) spin structure. Mathematicians call them the bounding and not-bounding spin structures, respectively.
- On a Riemann surface of genus g, there are 2^{2g} spin structures. They can be classified into even and odd ones, according to the number of zero modes of the Dirac operator. But for $g > 1$ there is no natural way to associate them with elements of $H^1(M, \mathbb{Z}/2)$.

Spin and statistics

- The relation between spin and statistics in lattice models is far from obvious.
- The lattice breaks $SO(d)$-invariance, so it is not immediately clear how to define a spin structure on a lattice. The key observation is that $w_2(M)$ can be defined combinatorially, using a triangulation of M. More on this later.
- People working on lattice gauge theories and lattice QCD routinely work with lattice Dirac operators. They have avoided the issue of spin structure by working almost exclusively with cubic lattices and periodic boundary conditions.

Simple check: classification of invertible fermionic phases with symmetry G in 2D.

They are classified by elements of
$H^2(BG, \mathbb{R}/\mathbb{Z}) \times H^1(BG, \mathbb{Z}/2) \times H^0(BG, \mathbb{Z}_2)$.
Fidkowski and Kitaev, 2010; Chen, Gu, and Wen, 2011

The last factor $H^0(BG, \mathbb{Z}/2) = \mathbb{Z}/2$ classifies invertible fermionic phases without any symmetry. The nontrivial element corresponds to a 1d system with a single Mayorana zero mode at each edge.

The group structure is a bit non-obvious:

$$(\alpha, \beta, \gamma) + (\alpha', \beta', \gamma') = (\alpha + \alpha' + \frac{1}{2}\beta \cup \beta', \beta + \beta', \gamma + \gamma').$$

Spin-TQFTs in 2D

(Unitary) spin-TQFTs in 2D have been studied by Moore and Segal, 2006, for the case of trivial G. There is a unique nontrivial invertible spin-TQFT.

The partition function $Z(M,\eta) = (-1)^{\mathrm{A}rf(M,\eta)}$. Here

$$\mathrm{A}rf(M,\eta) = \dim \ker D_L \mod 2.$$

where D_L is the left-handed Dirac operator.

The vector space $V(S^1,\eta)$ is one-dimensional for both choices of η. It is even for anti-periodic (NS) spin structure and odd for periodic (R) spin structure.

The category $\mathcal{C}(pt)$ is the category of modules over the algebra $Cl(1)$ (single odd generator θ satisfying $\theta^2 = 1$).

G-equivariant spin-TQFTs in 2D

If G is nontrivial, can appeal to a mathematical result (Freed, Hopkins): invertible unitary equivariant TQFTs are classified by the (dual of torsion part of) spin-bordism of BG.

The Atiyah-Hirzebruch spectral sequence collapses at E_2 and gives $H^2(BG, \mathbb{R}/\mathbb{Z}) \times H^1(BG, \mathbb{Z}/2) \times H^0(BG, \mathbb{Z}_2)$. The group structure is non-trivial, as above.

Let me describe $Z(M, A, \eta)$ for general G. Here $A : \pi_1(M) \to G$ describes a gauge field on M. Can also think of A as a G-valued function on 1-simplices of M.

Spin structures in 2D and quadratic refinements

Let M be a closed oriented 2D manifold. Its topology is determined by $g = \frac{1}{2} \dim H^1(M, \mathbb{Z}/2)$.

On $H^1(M, \mathbb{Z}/2)$ there is a non-degenerate bilinear form called the intersection form:

$$b(x, y) = \int_M x \cup y \in \mathbb{Z}/2.$$

A quadratic refinement of b is a function $q : H^1(M, \mathbb{Z}/2) \to \mathbb{Z}_2$ satisfying

$$q(x + y) - q(x) - q(y) = b(x, y).$$

It turns out there is a 1-1 correspondence between spin structures on M and quadratic refinements of b (Atiyah, 1971). Let q_η be the quadratic function corresponding to η.

The 2D partition function

We have a gauge field A on M. If we triangulate M, A gives an element of G for every 1-simplex. $\beta \in H^1(BG, \mathbb{Z}/2)$ is a function $\beta : G \to \mathbb{Z}/2$ satisfying $\beta(g_1 g_2) = \beta(g_1) + \beta(g_2)$ (i.e. a homomorphism from G to $\mathbb{Z}/2$). Then $\beta(A)$ is a $\mathbb{Z}/2$-valued gauge field on M. Call it $\beta_A \in Z^1(M, \mathbb{Z}/2)$.

Mathematically, β_A is a pull-back of $\beta \in Z^1(BG, \mathbb{Z}/2)$ to M using a map $A : M \to BG$.

Similarly, $\alpha \in Z^2(BG, \mathbb{R}/\mathbb{Z})$ pulls back to a $\alpha_A \in Z^2(M, \mathbb{R}/\mathbb{Z})$ ("flat B-field on M"). Then:

$$Z(M, A, \eta)_{\alpha,\beta,\gamma} = \exp\left(2\pi i \int_M \alpha_A\right) (-1)^{q_\eta(\beta_A)} (-1)^{\gamma \mathrm{A}rf(M,\eta)}.$$

Fermionic state-sum models

To explain the coincidence, we would like construct lattice models corresponding to spin-cobordism classes.

Bosonic state-sum models involve variables living on simplices of various dimensions and a weight ("action") which depends on these variables. The partition function is obtained as an "integral" of the weight over all allowed configurations of variables.

Fermionic state-sum models may also involve Grassmann variables. The partition function involves a Berezin integral over these variables.

The Dirac action for free fermions is an example of a non-topological fermionic state-sum model. We would like to construct models which are quasi-topological (depend only on the spin structure).

The Gu-Wen construction I

Let me focus on the case $\alpha = \gamma = 0$. The general case will follow easily once we understand this special case.

We need to construct $q_\eta(\beta_A)$, where η is a spin structure on M and $\beta_A \in Z^1(M, \mathbb{Z}/2)$. From now on abbreviate β_A to β.

We choose a triangulation of M, then β is a $\mathbb{Z}/2$-valued function on 1-simplices.

On each 1-simplex e with $\beta(e) = 1$ we place a pair of Grassmann variables $\theta_e, \bar{\theta}_e$. The integration measure is

$$\prod_{e, \beta(e)=1} d\theta_e d\bar{\theta}_e .$$

The Gu-Wen construction II

The integrand s a product over all 2-simplices P. To write down the contribution of a given 2-simplex, need to choose a local order on vertices (branching structure).

We assume M is oriented, so each P is oriented. A branching structure also gives an orientation to each P, so we have two kinds of 2-simplices ($+$ and $-$).

Each e is shared by a $+$ triangle and a $-$ triangle. If $\beta(e) = 1$, we assign θ_e to the $+$ side and $\bar{\theta}_e$ to the $-$ sign.

The weight of a triangle P is a product of all Grassmann variables on its edges, taken in the order specified by the global orientation. The weight of each P is even.

The result of the integration is a sign $\sigma(\beta)$ which depends on the cocycle β, the triangulation, and the branching structure.

The Gu-Wen construction III

To fix this, we need a correction factor (D.Gaiotto and A.K., 2015). For simplicity, let the triangulation be the barycentric subdivision of another triangulation, then there is a canonical branching structure. Still need to worry about triangulation and choice of β within its cohomology class.

The correction factor is

$$\prod_{e \in E} (-1)^{\beta(e)},$$

where E is a 1-chain satisfying $\partial E = \sum v$.

The chain E defines a spin structure on M: $\sigma(\beta)$ multiplied by the correction factor is a quadratic function of $[\beta] \in H^1(M, \mathbb{Z}/2)$ which refines the intersection pairing.

Spin-statistics relation on a lattice

We can regard the 1-chain E as a 1-cochain η on the dual cell complex. Then the correction factor becomes

$$(-1)^{\int_M \beta \cup \eta}.$$

The 2D case exemplifies a general principle: to construct a weight out of Grassmann variables one needs to choose an order on vertices. To eliminate the dependence on this order, one needs a correction factor.

The correction factor depends on a cochain $\eta \in C^1(M, \mathbb{Z}/2)$ such that $\delta\eta = w_2$, where $w_2 \in Z^2(M, \mathbb{Z}/2)$ represents the 2nd Stiefel-Whitney class.

The 1-cochain η is the lattice version of the spin structure. In the 2D case, the Poincare dual of w_2 is $\sum v$, so that the 1-cycle E is the Poincare dual of η.

The Arf invariant as a state-sum

The Arf invariant $\mathrm{A}rf(M,\eta)$ can be expressed through q_η:

$$(-1)^{\mathrm{A}rf(M,\eta)} = 2^{-g(M)} \sum_{\beta\in H^1(M,\mathbb{Z}/2)} (-1)^{q_\eta(\beta)}.$$

This can be written as a state-sum too: apart from the Berezin integral, one also has $\mathbb{Z}/2$-valued variables $\beta(e)$ on 1-simplices satisfying the constraint $\delta\beta = 0$.

This state-sum can be thought of as a topological $\mathbb{Z}/2$ gauge theory, with the gauge field β determining where the Grassmann variables $\theta, \bar{\theta}$ live.

Thus $Z(M, A, \eta)_{\alpha,\beta,\gamma}$ admits a fermionic state-sum construction for any α, β, γ.

State-sum construction of 2D oriented TQFTs

Fukuma, Hosono, Kawai, 1991

Input: semi-simple algebra A with a symmetric non-degenerate scalar product $A \otimes A \to \mathbb{C}$ satisfying $\langle a, bc \rangle = \langle ab, c \rangle$ (i.e. A is a semi-simple Frobenius algebra).

Construction: choose an orthonormal basis $e_i \in A$. Edges are labeled by e_i. Weights: $C_{ijk} e_i \otimes e_j \otimes e_k$ for a 2-simplex P whose sides are labeled by e_i, e_j, e_k. Then contract across edges using the scalar product.

This gives a TQFT because (1) C_{ijk} are cyclically-symmetric; (2) define associative multiplication.

Space attached to a circle $V(S^1)$ is the center of A.

State-sum construction of 2D spin TQFTs

D. Gaiotto and A. K. 2015

Input: semi-simple Frobenius algebra A equipped with a $\mathbb{Z}/2$-grading. The scalar product is assumed symmetric and even.

Construction: the same thing, but take into account the Koszul sign arising from permuting the factors in the tensor product before contracting.

The Koszul sign is $\sigma(\beta)$, with $\beta(e)$ telling us whether the edge e is labeled by a bosonic or a fermionic element of A.

Inserting the correction factor

$$\prod_{e \in E} (-1)^{\beta(e)} = (-1)^{\int_M \beta \cup \eta},$$

get a triangulation-independent partition function.

Bosonization in 2D

The same data ($\mathbb{Z}/2$-graded semi-simple algebra) can be used to construct a TQFT with symmetry $\mathbb{Z}/2$. This leads to a correspondence between bosonic phases with $\mathbb{Z}/2$-symmetry and fermionic phases with no symmetry (apart from $(-1)^F$).

This might seem surprising: there are no nontrivial invertible bosonic phase with $\mathbb{Z}/2$ symmetry, but there is a nontrivial invertible spin-TQFT with $Z(M,\eta) = (-1)^{\mathrm{A}rf(M,\eta)}$.

Resolution: the nontrivial invertible fermionic phase is mapped to the trivial bosonic phase with $\mathbb{Z}/2$ symmetry, while the trivial fermionic phase is mapped to a bosonic phase with a spontaneously broken $\mathbb{Z}/2$.

Bosonization in 2D, cont.

In general, the correspondence is easy to describe on the level of partition functions.

A bosonic partition functions $Z_b(M, \beta)$ depends on a gauge field $\beta \in H^1(M, \mathbb{R}/\mathbb{Z})$. A fermionic partition function $Z_f(M, \eta)$ depends on a spin structure $\eta \in Spin(M)$.

The map and its inverse are

$$Z_f(M, \eta) = 2^{-g(M)} \sum_{\beta \in H^1(M, \mathbb{Z}/2)} Z_b(M, \beta)(-1)^{q_\eta(\beta)}.$$

$$Z_b(M, \beta) = 2^{-g(M)} \sum_{\eta \in Spin(M)} Z_f(M, \eta)(-1)^{q_\eta(\beta)}.$$

Invertible fermionic phases in 3D with symmetry G

In 2+1D, invertible spin-TQFTs with symmetry G are classified by the dual of 3D spin-bordism of BG. This group can be described as the space of solutions of the equation

$$\delta\alpha = \frac{1}{2}\beta \cup \beta + \frac{1}{4}\mathfrak{P}(\gamma \cup \gamma),$$

where $\alpha \in C^3(BG, \mathbb{R}/\mathbb{Z})$, $\beta \in Z^2(BG, \mathbb{Z}/2)$, $\gamma \in Z^1(BG, \mathbb{Z}/2)$ (modulo some complicated equivalence relation). $\mathfrak{P}$ is the Pontryagin square.

For example, for $G = \mathbb{Z}/2$, the space of solutions is $\mathbb{Z}/8$. This agrees with the result of Gu and Levin.

For $\gamma = 0$, there is a fermionic state-sum construction of the corresponding partition function $Z(M, A, \eta)_{\alpha,\beta}$. It uses a 3D version of the Gu-Wen Grassmann integral.

The Gu-Wen construction in 3D

One uses the gauge field A on M to produce a $\alpha_A \in Z^3(M, \mathbb{R}/\mathbb{Z})$ and $\beta_A \in Z^2(M, \mathbb{Z}/2)$. Grassmann variables $\theta_P, \bar{\theta}_P$ live on 2-simplices P for which $\beta_A(P) = 1$.

The weight is a product of weights associated with 3-simplices. Each 3-simplex T contributes $\exp(2\pi\alpha_A(T))$ times a monomial in $\theta, \bar{\theta}$.

The result of fermionic integration is

$$Z_{naive} = \exp(2\pi i \int_M \alpha_A)\sigma(\beta_A),$$

where $\sigma(\beta_A)$ is a sign. It depends on the branching structure and triangulation. Z_{naive} may also change sign if one replaces (α, β) by an equivalent one.

This can be fixed using a correction factor $\prod_{P \in E}(-1)^{\beta(P)}$, where $E \in Z_2(M, \mathbb{Z}/2)$ satisfies $\partial E = \sum e$. E is the dual of a spin structure.

State-sum construction of 3D oriented TQFTs

Turaev and Viro, 1992

Input: spherical fusion category $\mathcal{C}$.

The partition function is a sum over labelings of a triangulation. 1-simplices are labeled by simple objects E_i of $\mathcal{C}$, 2-simplices are labeled by basis vectors of $\mathrm{Hom}(\mathbf{1}, E_i \otimes E_j \otimes E_k)$.

The weight of a labeled triangulation is, roughly, a product over 3-simplices. Each 3-simplex contributes a 6j symbol.

State-sum construction of 3D spin TQFTs

Gu, Wang, and Wen, 2010; Gaiotto and A. K. 2015.

Input: spherical super-fusion category.

This means that morphism spaces are $\mathbb{Z}/2$-graded. In particular, for any simple object E_i the endomorphism space is a super-division algebra, i.e. either $\mathbb{C}$ or $Cl(1)$.

If for every simple E_i we have $Hom(E_i, E_i) = \mathbb{C}$, then the construction is the same as above, except that the weight is multiplied by the Gu-Wen sign $\sigma(\beta)$ and the correction factor $\prod_{P \in E}(-1)^{\beta(P)}$. Here $\beta \in Z^2(M, \mathbb{Z}/2)$ tells us which 2-simplices are labeled by fermionic basis elements.

The general case (when $Hom(E_i, E_i) = Cl(1)$ for some i) is not completely understood.

Bosonization in 3D

There is a 1-1 correspondence between 3D fermionic phases and 3D bosonic phases with an anomalous 1-form $\mathbb{Z}/2$-symmetry.

A parameter of a global 1-form $\mathbb{Z}/2$ symmetry is a $\mathbb{Z}/2$-valued 1-cocycle (i.e. a $\mathbb{Z}/2$ gauge field). Gauging means coupling the system to a 2-form gauge field $B \in Z^2(M, \mathbb{Z}/2)$.

The symmetry is anomalous in the sense that the partition function is not invariant under $B \mapsto B + \delta\lambda$, $\lambda \in C^1(M, \mathbb{Z}/2)$. Instead:

$$Z_b(B + \delta\lambda) = Z_b(B) \exp(\pi i \int_M (\lambda \cup \delta\lambda + \lambda \cup B + B \cup \lambda)).$$

Bosonization in D dimensions

It appears that in D dimensions there is a relation between fermionic phases and bosonic phases with an anomalous $(D-2)$-form $\mathbb{Z}/2$ symmetry.

The partition function of the bosonic theory depends on a $(D-1)$-cocycle $\beta \in Z^{D-1}(M, \mathbb{Z}/2)$.

The anomaly is controlled by a $(D+1)$-dimensional topological action $\int_X Sq^2 \beta$, where Sq^2 is the Steenrod square operation.

Some open questions

- What does a fermionic state-sum construction look like for a general super-fusion category?
- For $D > 3$, not every oriented manifold is spin. How does this affect bosonization?
- Can one prove a spin-statistics relation for unitary lattice models in all dimensions?
- What is the "right" lattice version of the Dirac operator, so that it depends on a lattice spin structure?