

1 Speed of light is the maximal possible speed

We saw that the length of a moving object shrinks. If its length at rest was l_0 , then when it is moving with velocity v its length becomes

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

But what if $v = c$? Then this formula gives $l = 0$, which looks strange. If $v > c$, even worse things happen: $1 - v^2/c^2$ becomes less than zero, and then its square root does not exist! (or becomes purely imaginary, if you know what I am talking about).

This suggests that perhaps it is not possible for any object to move faster than light. And perhaps it is not possible to accelerate a slowly moving object to the speed of light. (One might even be tempted to say that light is the only thing which can move with the speed of light. But it is now known that other things can do this too. For example, gravitational waves also move with the speed of light.)

2 Adding velocities

One can try to disprove this conjecture by imagining the following situation. Suppose a spaceship moves past Earth with the speed $2c/3$. Let us call it spaceship A. Spaceship B moves in the same direction and overtakes spaceship A. Suppose the astronaut in spaceship A measures the speed of spaceship B and finds that it is $2c/3$ (with respect to spaceship A). Then what is the speed of spaceship B with respect to the Earth? Isn't it $2c/3 + 2c/3 = 4c/3$? This would be larger than c !

The resolution of this paradox is that from the point of view of the observer on the Earth the difference between the velocities of spaceships B and A is less than $2c/3$. Indeed, suppose the astronaut in spaceship A uses his own spaceship as a ruler. He places stopwatches at the two ends of his spaceship and marks the time that it takes the nose of spaceship B to get from the poop to the nose of spaceship A. (What is poop doing here? Well, this is what the back part of a sailing ship used to be called.) Suppose this time is T_0 . He also knows that the length of spaceship A is L_0 , and concludes that the velocity of spaceship B is L_0/T_0 .

Now let us examine this measurement from the point of view of the observer on the Earth. From her point of view, the length of spaceship A is smaller than L_0 . And the moving stopwatches slow down compared to watches left on the Earth, so the time measured by the watches on the Earth is longer than T_0 . So the velocity of B relative to A from the point of view of the observer on the Earth will be smaller than L_0/T_0 .

Actually, there is yet another effect which needs to be taken into account. The astronaut in spaceship A is sure that both of his stopwatches are properly synchronized. That is, they show time 0:00:00 at the same moment. But this is not so from the point of view of the observer on the Earth. She thinks that it is the stopwatch at the poop which first shows time 0:00:00, and a bit later the stopwatch at the nose shows the same time. (We already discussed how this comes about.) Hence according to the observer on the Earth the stopwatch at the nose is a bit behind the stopwatch at the poop. This time lag between the two stopwatches has to be added to the results of the time measurements. (Think about it like this: if the stopwatch at the nose is behind by 1 second, and according to this stopwatch the nose of spaceship B caught up with the nose of spaceship A at 12:00:00 pm, then it actually caught up at 12:00:01.) So this third effect works in the same direction as the first two effects (time dilation and length contraction) and decreases the relative velocity of B and A as measured by the observer on the Earth.

What I said is enough to work out the velocity of spaceship B relative to the Earth. Since the algebra is a little involved, I will not show this computation and simply state the result. If spaceship A moves with velocity v_A with respect to the Earth, and the astronaut on spaceship A measures the velocity of spaceship B to be v_B , then the observer on the Earth will measure the following velocity for spaceship B:

$$\frac{v_A + v_B}{1 + \frac{v_A v_B}{c^2}}$$

This is called the relativistic law of addition of velocities. For example, if $v_A = v_B = 2c/3$, then according to this formula the observer on the Earth will find that the velocity of spaceship B is $12c/13$ which is less than c . So perhaps our conjecture that the speed of light is the maximal possible speed is true.

3 Mass and momentum of a moving object

Here is another attempt to disprove the conjecture. According to Newton's second law, if a force F acts on an object of mass m , it accelerates, and the acceleration a is determined from

$$F = ma.$$

Suppose F is constant (i.e. does not change) for a very long time. Then the acceleration stays constant for a very long time. The velocity of an object will grow at a constant rate, and given enough time will become arbitrarily large. Larger than c , or $10c$, or even $10^{100}c$.

The loophole in this argument is that the mass of a moving object actually increases as the velocity approaches c . So the acceleration becomes smaller and smaller. The velocity of an object never reaches c because of this effect.

How do we know that the mass of a moving object really increases? I will explain this in a second, but first a quick reminder about the notion of *momentum*. Momentum of an object is defined as $p = mv$. The intuitive meaning of momentum is "punch". If something hits you, the force that you will feel will be determined by the momentum of the thing which hit you. This is because the second law of Newton can also be written as follows:

$$F = m \frac{v_2 - v_1}{t_2 - t_1} = \frac{p_2 - p_1}{t_2 - t_1}.$$

Here $p_1 = mv_1$ is the initial momentum of an object (at time t_1), $p_2 = mv_2$ is the final momentum of an object (at time t_2). Thus force is equal to the rate of change of momentum of an object. If something hits you, it stops and gives its momentum to you. Your momentum changes from zero to something; and this change in your momentum is what creates the force of a punch.

This new form of Newton's law is better than the original one because it applies even when mass changes with time. Indeed, imagine that you are pushing a shopping cart in a store, and your mom keeps putting more and more food into the cart. It seems obvious that to keep the cart moving at a constant speed you will have to exert force. This seems to contradict Newton's second law $F = ma$: $a = 0$, but F is nonzero! This is because the law is wrong in this situation. The new form of Newton's law is OK though. Even though the velocity v of the cart does not change, its mass m increases, so the momentum of the cart increases. If the momentum of the cart increases, someone (you) must push it with a nonzero force.

OK, now back to relativity theory. Imagine two parallel train tracks, and two trains A and B traveling in opposite directions on the two tracks. On the train A there is a passenger named Alex. On the train B there is his identical twin brother Boris. They are standing on the roofs of their carriages, and as the carriages pass by each other, they punch each other, with equal strength.

Suppose the velocities of both trains are very large (close to c) and equal. What will Alex see? Well, he will see Boris' fist moving very slowly, because *everything* that Boris does will seem slow to him (time dilation!) Then the momentum of Boris' fist will be very small, much smaller than that of Alex's fist, right? And Alex's fist will push Boris' fist back to him.

But wait! From Boris' point of view everything looks the other way around. It is Alex who moves very slowly, while from Boris' point of view he moves with the usual speed. So it is Boris' punch which should be stronger.

Both of these accounts cannot be true. Moreover, from the point of view of the observer at rest it is clear that the situation is completely symmetric and Alex will punch Boris just as strongly as Boris will punch Alex. The only way out of a contradiction is to say that although from Alex's point of view the velocity of Boris' fist is smaller than Alex's, its mass is larger by the same amount, so the momentum of Boris' fist is the same as the momentum of Alex's fist.

By how much does Boris' fist slow down because of time dilation? By a factor

$$\sqrt{1 - \frac{v^2}{c^2}}.$$

Hence the mass should increase by the same factor. If m_0 is the mass of Boris' fist at rest, then when it is moving the mass must become

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

m_0 is known as the rest mass.

Then the momentum of the moving fist is

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Although we established this formula in a special situation, one can show that it is completely general. Mass of any moving object increases in the same proportion. Indeed, if it did not, then two objects which balance each

other on scales when the scales are at rest would not balance each other any more when the scales are moving at a constant speed. But this would contradict Galileo's relativity principle (there should be no way to establish whether you are at rest or moving at a constant speed).

There is another interpretation of the relativistic formula for momentum. One may agree once and for all to call m_0 *the* mass of an object and never mention the velocity-dependent mass m . Then one says that in relativity theory the momentum of an object is given not simply by m_0v , as in Newton's mechanics, but by the more complicated formula

$$p = \frac{m_0v}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

But it is all a matter of interpretation. I like to think about velocity-dependent mass m , since this is rather intuitive.