Stability Analysis of Thin-Walled Inflating Regions in de Sitter Space

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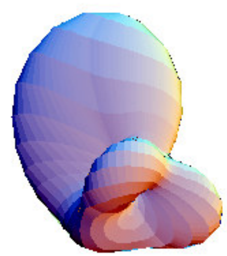


Acknowledgments

Thanks to:

- **Anthony Aguirre.**
- **Tom Banks, Michael Dine, and John Faulkner.**
- **Fellow grads: especially Trieu Mai and John Mason.**

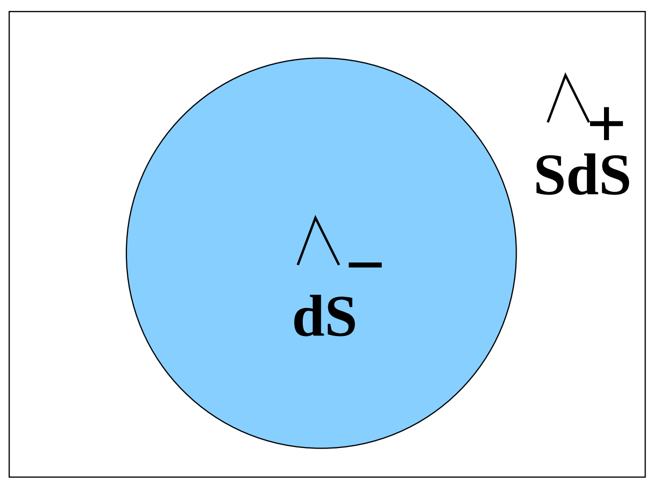
Outline and Conventions

- Junction Condition Formalism: Full GR.
- Perturbation analysis: Motion in a background.

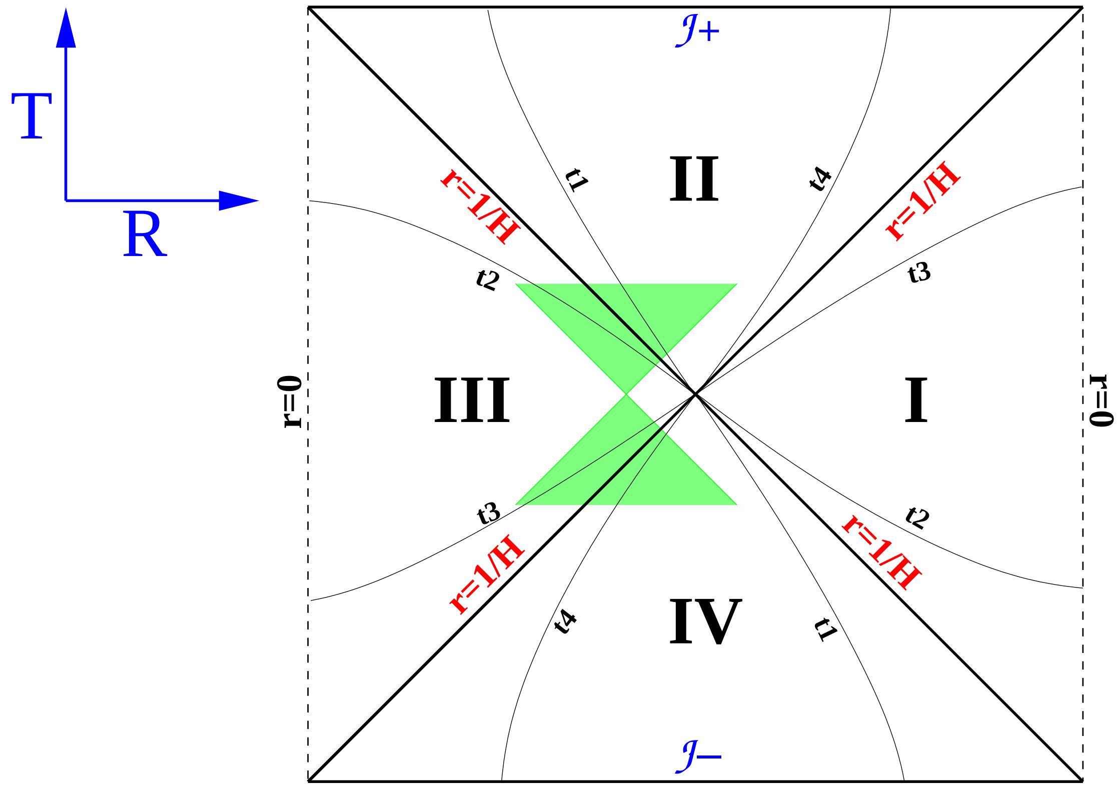
- Lorentzian metric signature diag(-1,1,1,1).
- Geometrical units $\hbar = c = G = 1$.

Junction Condition Formalism

- Israel: match two space times across an infinitesimally thin spherical shell of mass.
  - Require metric continuity across boundary.
  - Solve Einstein's equations.
- Guth et. al. (1987-1990) - Is it possible to create a universe in the lab?
- de Sitter space can make universes via fluctuations (DKS and AS).
  - Interior space time will be de Sitter with a cosmological constant Λ_-.
  - Exterior space time will be Schwarzschild de Sitter (spherically symmetric mass in de Sitter space) with a cosmological constant $\Lambda_+ < \Lambda_-$.



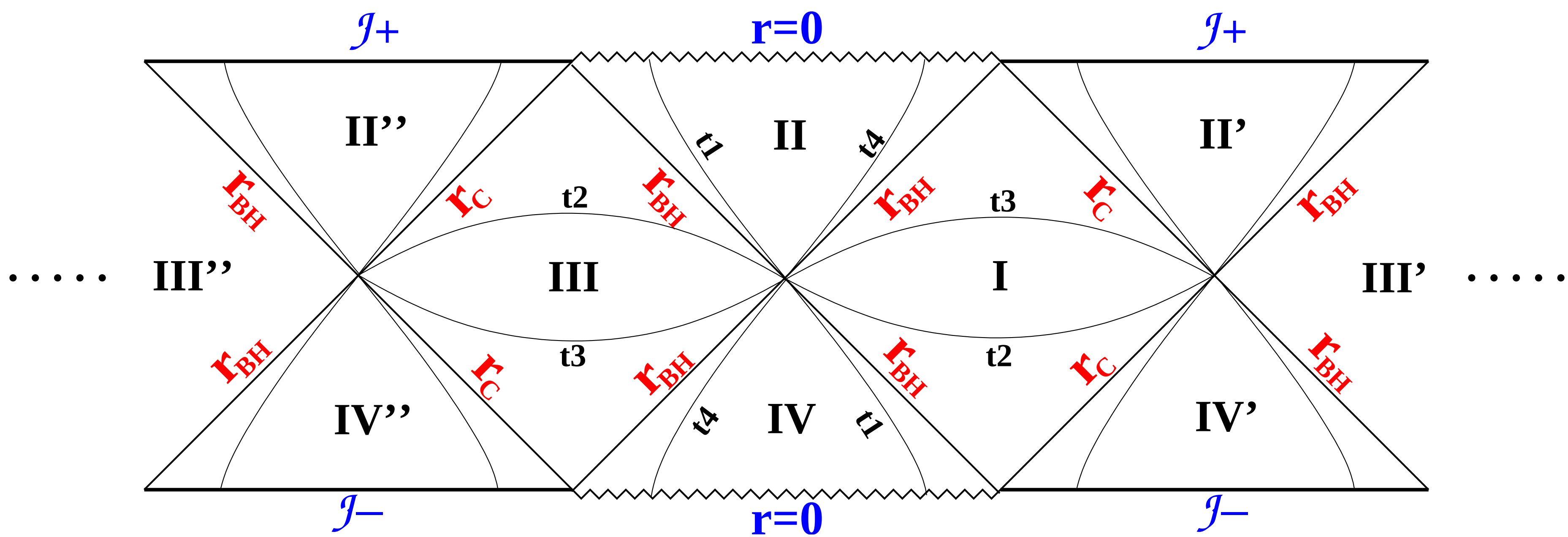
Interior de Sitter



$$ds_{-}^2 = -a_{ds}dt^2 + a_{ds}^{-1}dr^2 + r^2 d\Omega^2.$$

$$a_{ds} = 1 - \frac{\Lambda_-}{3}r^2; \quad H = \sqrt{\frac{\Lambda_-}{3}}$$

Exterior Schwarzschild de Sitter: $\Lambda_+ < 3M$



$$ds_+^2 = -a_{sds}dt^2 + a_{sds}^{-1}dr^2 + r^2 d\Omega^2,$$

$$a_{sds} = 1 - \frac{2M}{r} - \frac{\Lambda_+}{3}r^2.$$

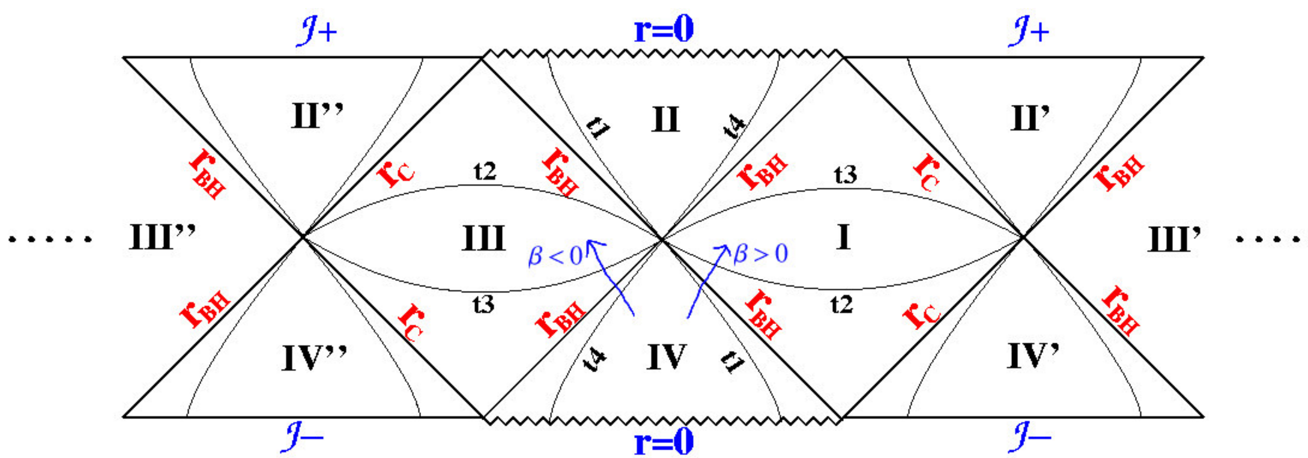
Thin Walled Bubbles

Bubble wall radius r is the only dynamical variable. The equation of motion for r is found to be (Guth et. al., Aurilia et. al.):

$$\beta_{ds} - \beta_{sds} = kr$$

$$\beta_{ds} = -a_{ds}\frac{dt_{ds}}{d\tau}\ , \qquad \beta_{sds} = a_{sds}\frac{dt_{sds}}{d\tau}\,.$$

$$k = 4\pi\sigma$$



Dimensionless Variables

Can define dimensionless variables to express the solution as motion in a 1-D potential with “energy” Q

$$\left[\frac{dz}{d\tau'}\right]^2 = Q - V(z).$$

$$z = \left(\frac{L^2}{2M}\right)^{\frac{1}{3}} r,$$

$$\tau' = \frac{L^2}{2k}\tau,$$

$$L^2 = \frac{1}{3}\left[\left|\left(\Lambda_-^2 + \Lambda_+^2 + 3k^2\right) - 4\Lambda_+\Lambda_-\right|\right]^{\frac{1}{2}}$$

$$Q = -\frac{4k^2}{(2M)^{\frac{2}{3}} L^{\frac{8}{3}}}$$ Note: Q small when M large.

Potential

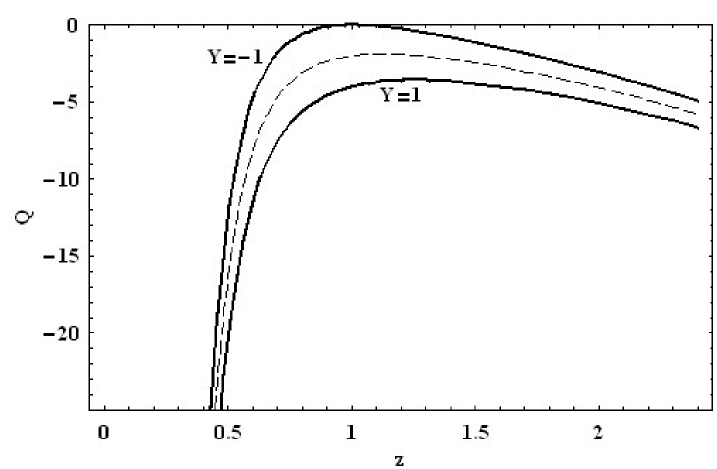
The potential $V(z)$ is given by:

$$V(z) = -\left[z^2 + \frac{2Y}{z} + \frac{1}{z^4}\right]$$

$$Y = \frac{1}{3}\frac{\Lambda_+ - \Lambda_- + 3k^2}{L^2}; \quad -1 \leq Y \leq 1.$$

Convenient to define A and B by

$$\Lambda_+ = Ak^2 \quad \Lambda_- = Bk^2$$



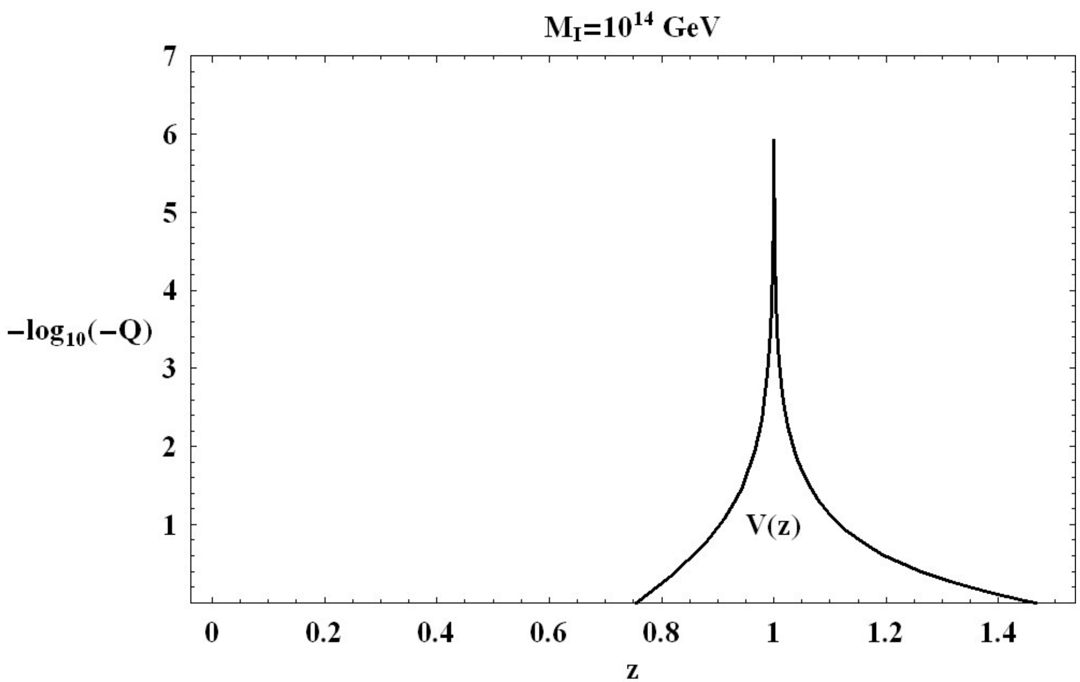
A Realistic Potential

Assume $M_I \simeq 10^{14}$ GeV and a very small fundamental cosmological constant.

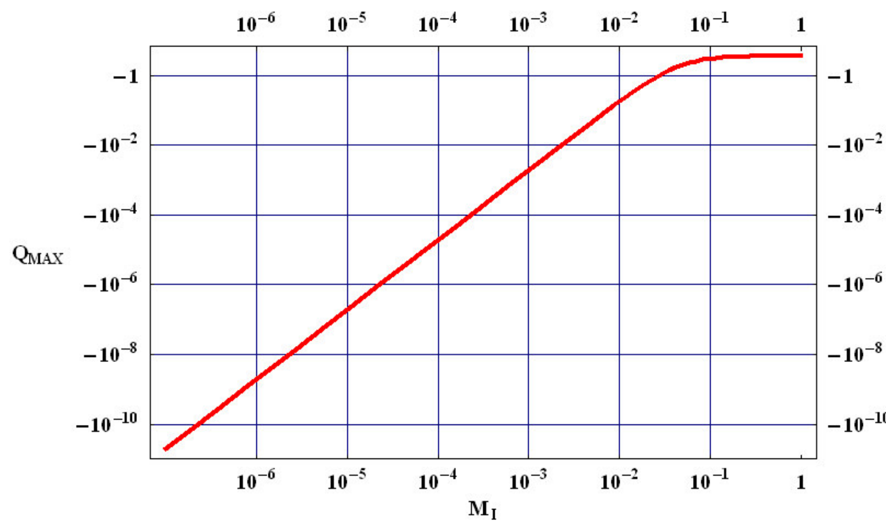
$$k \simeq 4\pi 10^{-15}$$

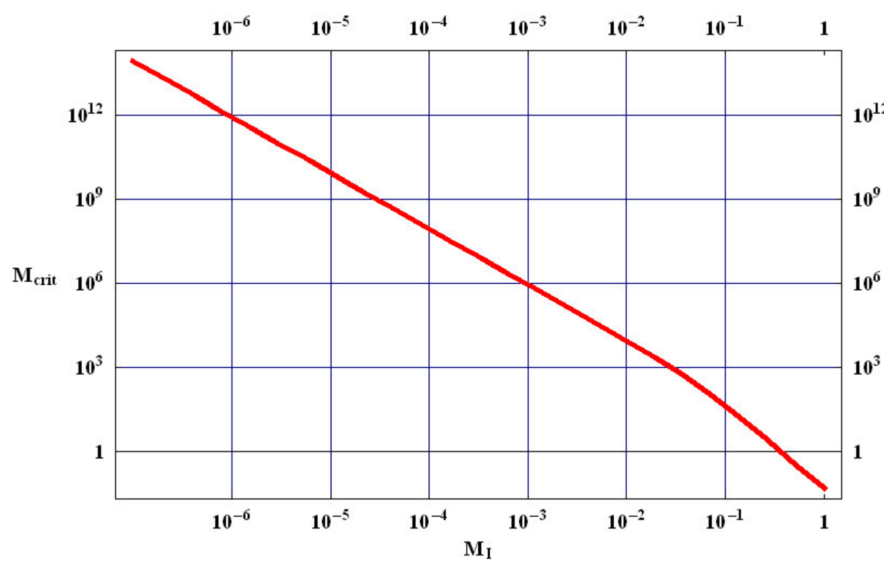
$$\Longrightarrow A \simeq 0 \qquad \Longrightarrow B \simeq 10^7$$

$$\Longrightarrow Y \simeq -1$$

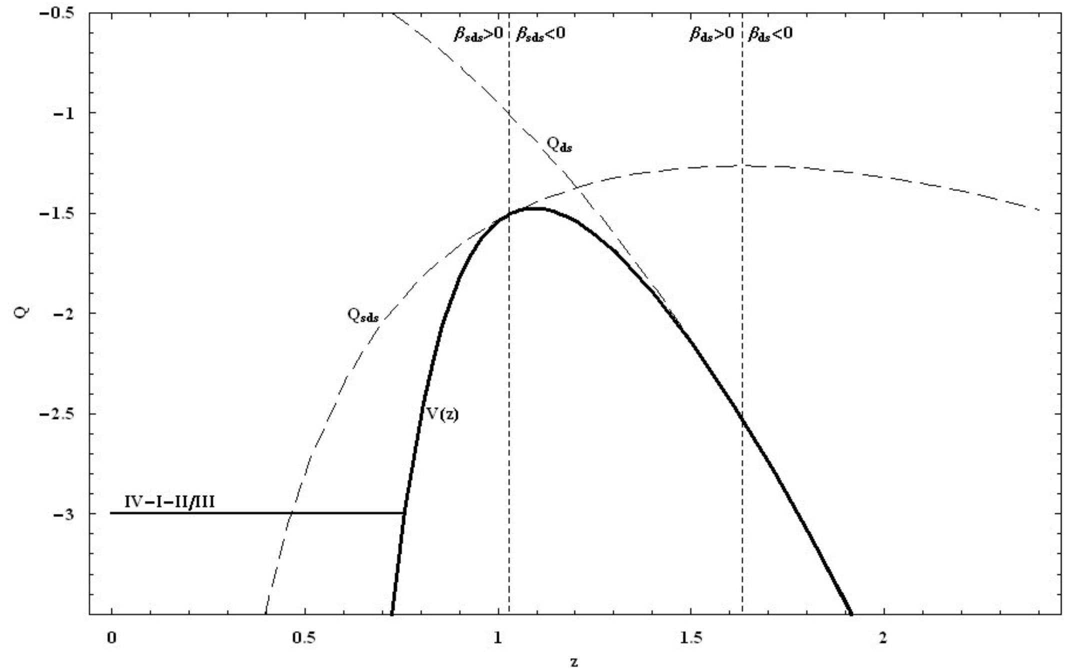


Relating Mass and Q *for* $A \simeq 0$

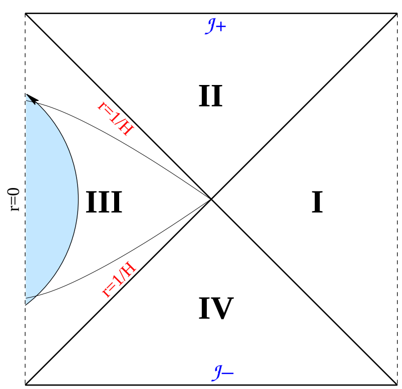




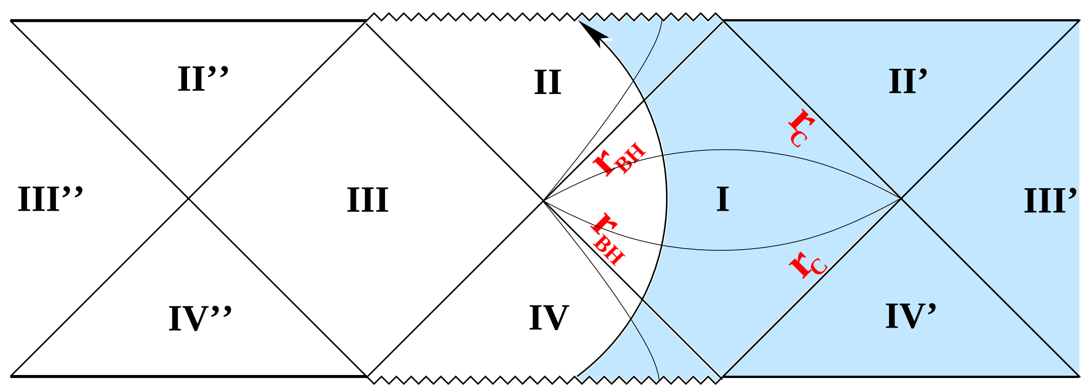
IV-I-II/III



Q
z
$\beta_{sds}>0$ $\beta_{sds}<0$
$\beta_{ds}>0$ $\beta_{ds}<0$
Q_{ds}
Q_{sds}
V(z)
IV−I−II/III

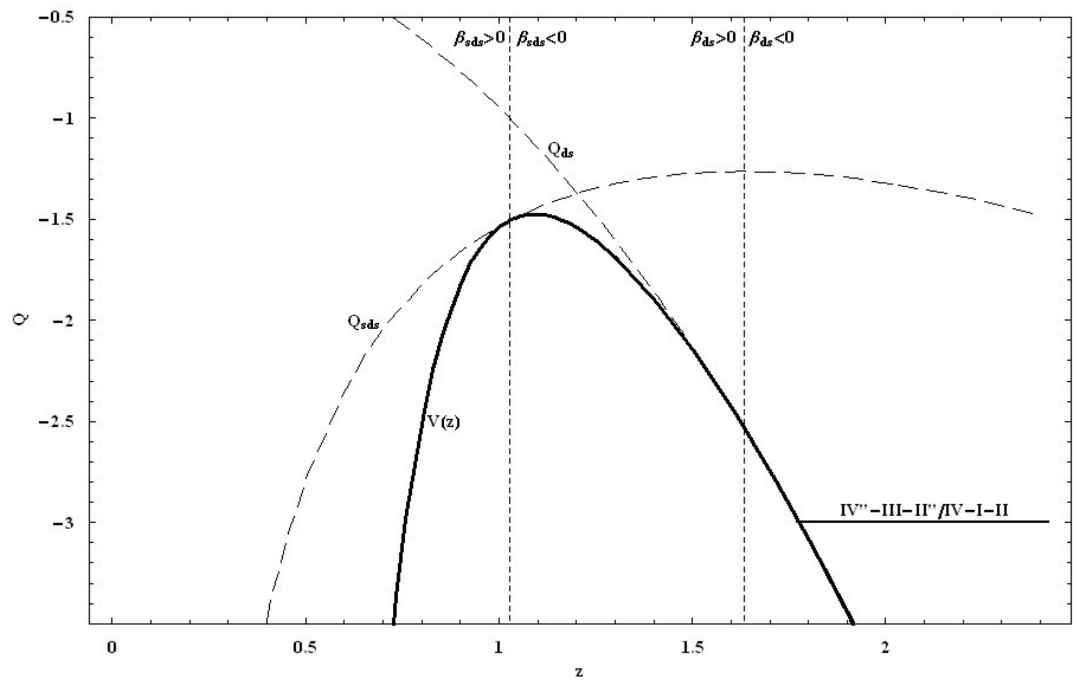


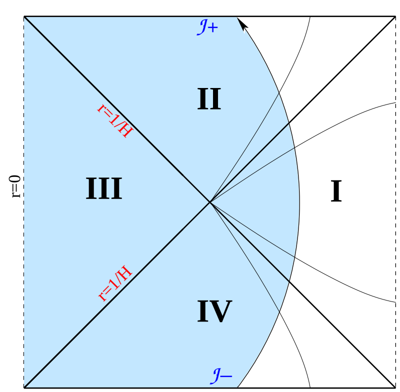
r=0
II
III
I
IV
r=1/H
r=1/H

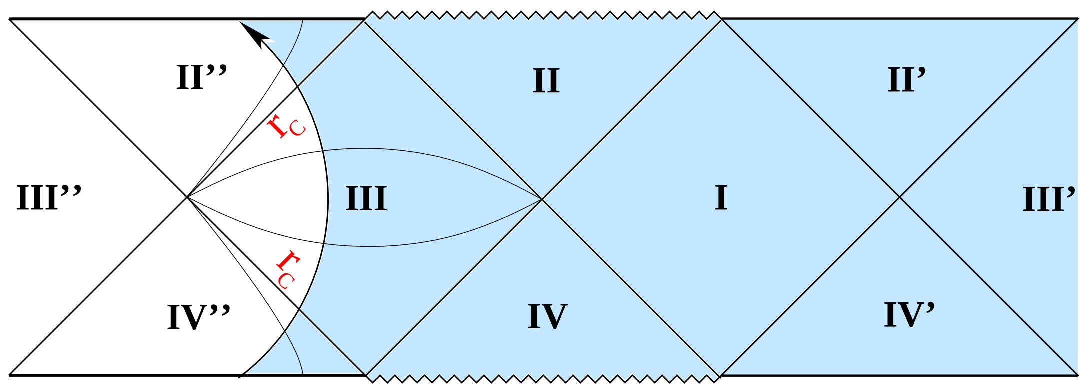


II''
III''
IV''
III
II
IV
I
II'
III'
IV'
r_{BH}
r_{BH}
r_C
r_C

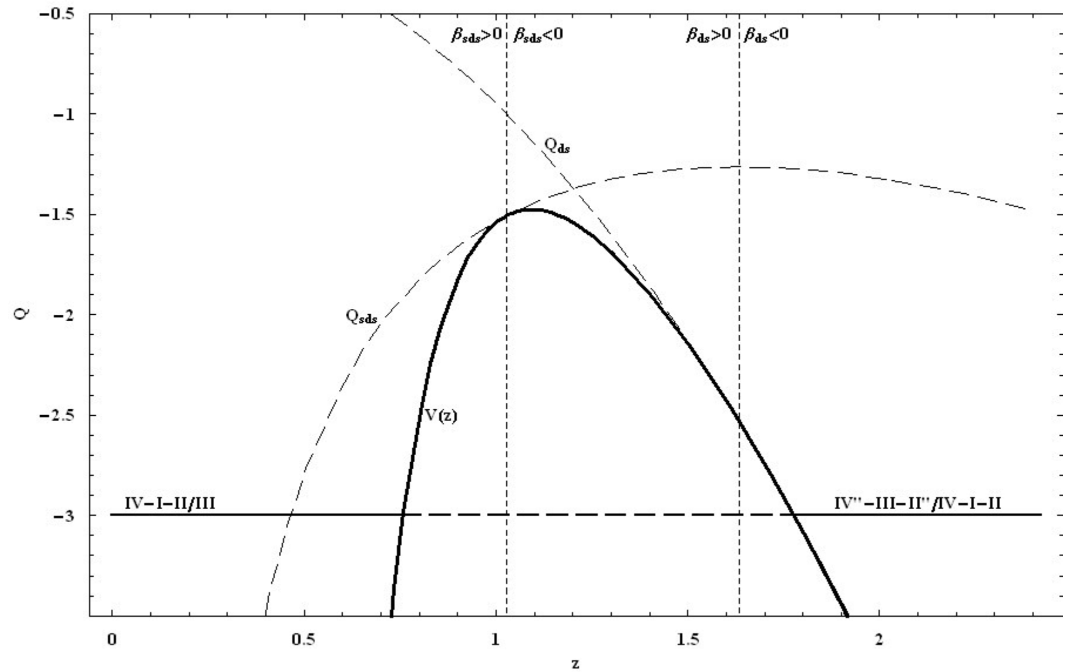
IV”-III-II”/IV-I-II

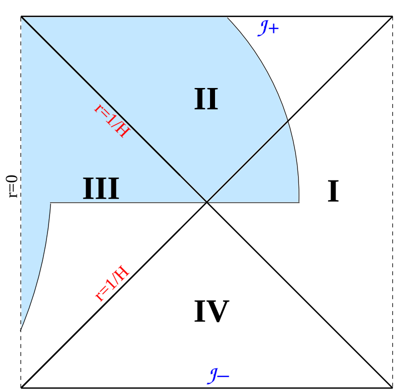


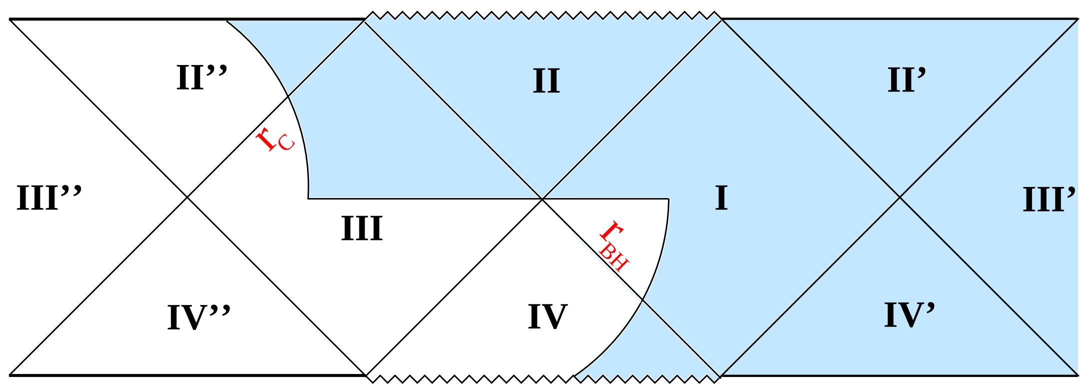




Can This Happen?





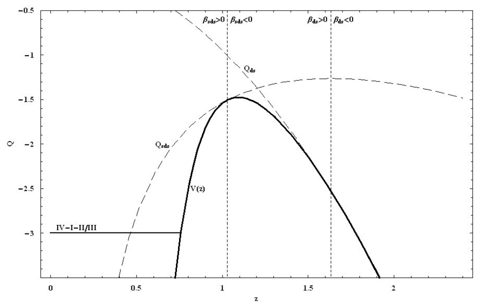


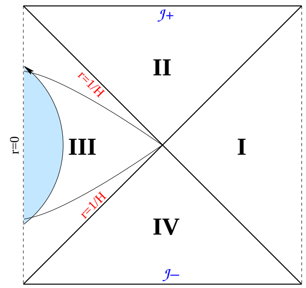
Tunneling

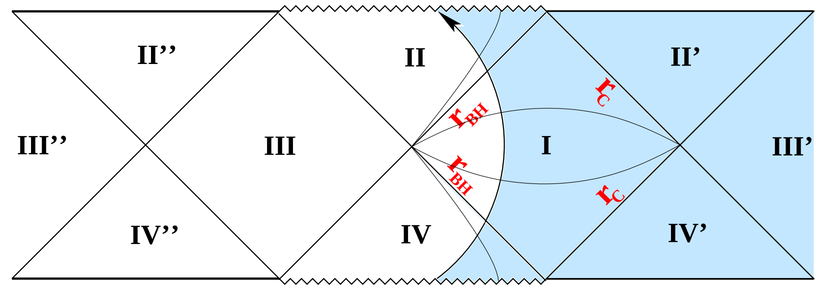
- Can a bound solution tunnel to an unbound solution (Farhi et. al., Fischler et. al.)?
- IV-I-II/III - the only classically buildable solution. Need tunneling to make a universe in the lab.
- In the context of semi-classical quantum gravity, tunneling occurs! But, we don't have full quantum gravity......
- In the process, we have created a baby universe!

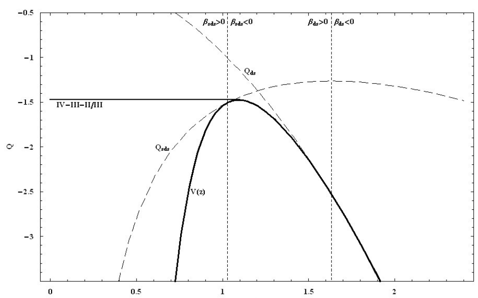
The only chance for this is at the turning point!

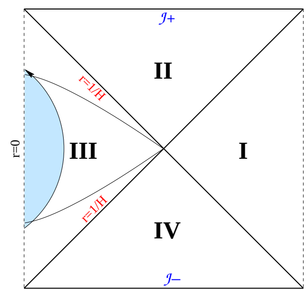
Bound Solutions

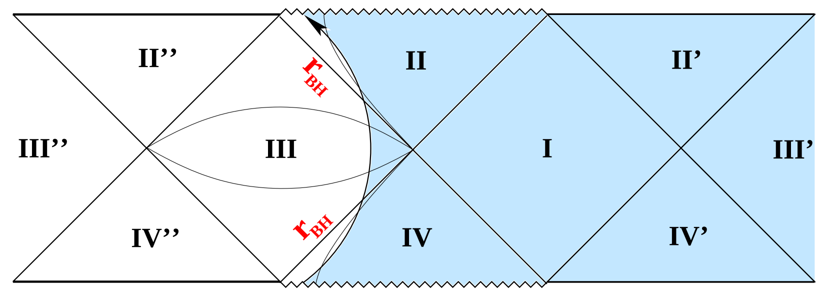




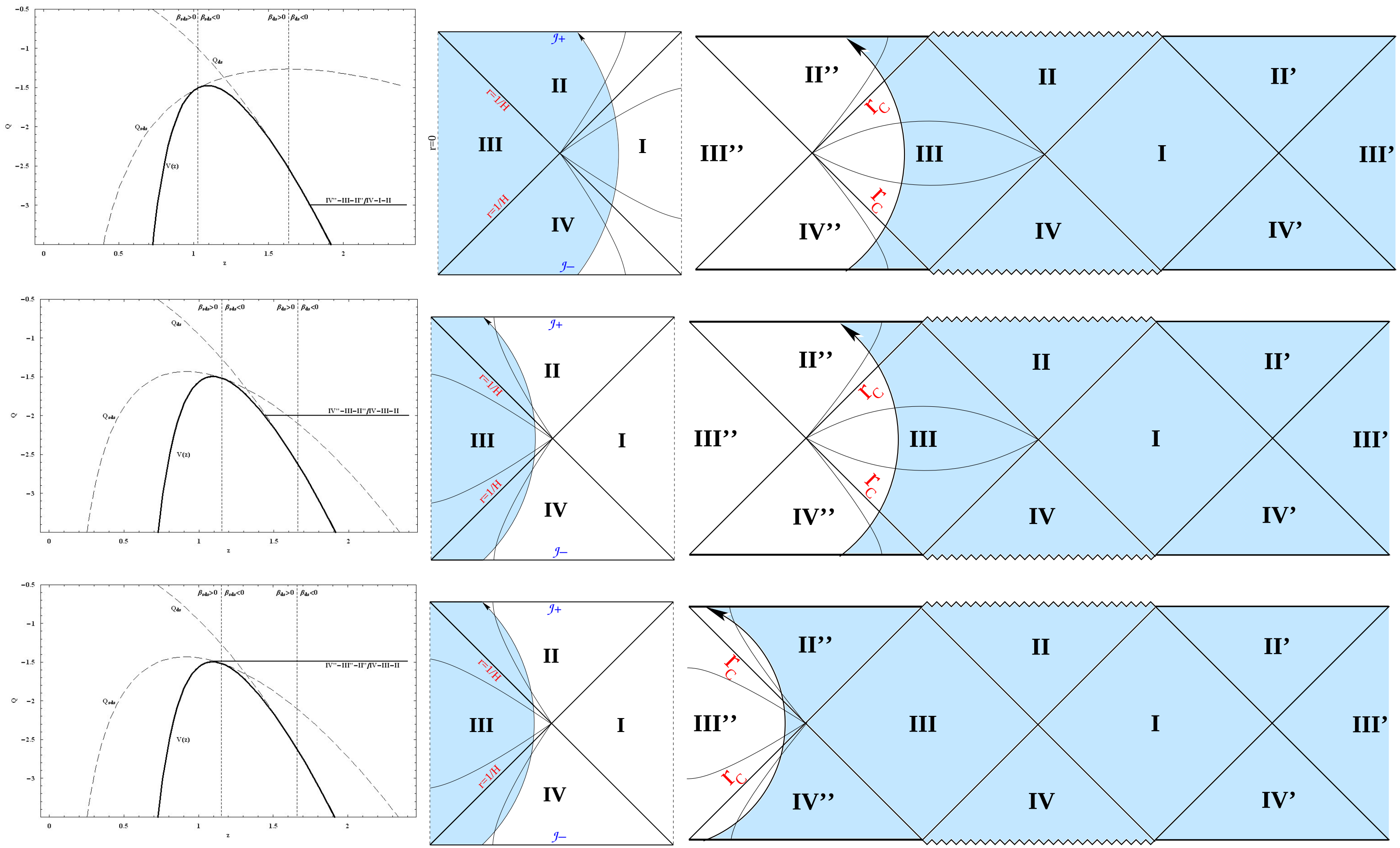




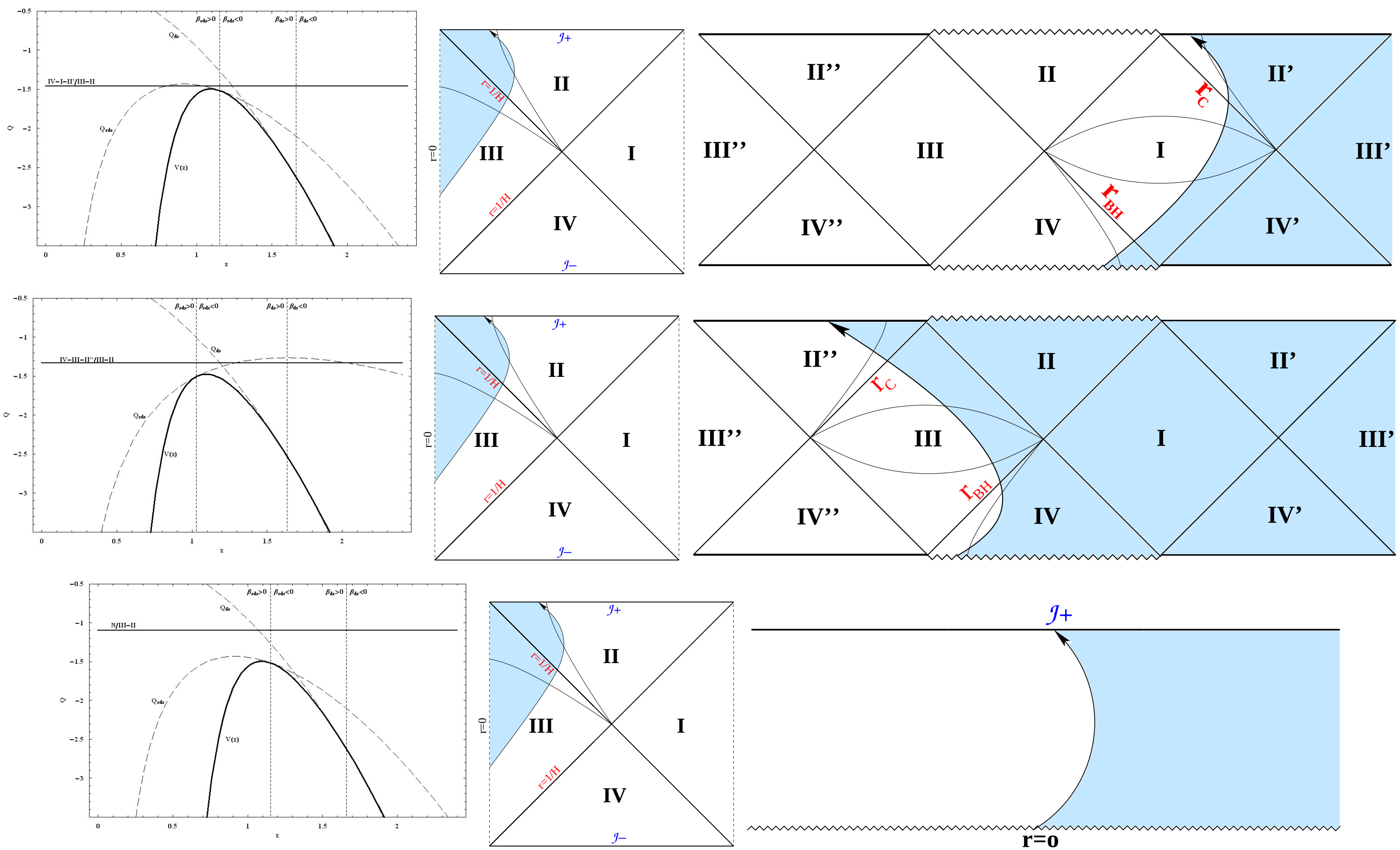




Unbound Solutions



Monotonic Solutions



Stability

- Have so far assumed spherical symmetry.
- This configuration is unstable!
  - Λ_+ is energetically favorable.
  - Divets grow as volume energy $\longrightarrow$ wall energy.
- Perturbations grow for all solutions, focus on bound solutions.
  - Relevant to the tunneling problem: Is the bubble spherically symmetric at the turning point?

Can use a simplified formalism to examine the stability!

Simplified Action in SdS: Motion in a Background

- Neglect gravitational contributions to action

$$S = -\sigma \int d^3\xi \sqrt{-\gamma} + \epsilon \int d^4x \sqrt{-g}$$

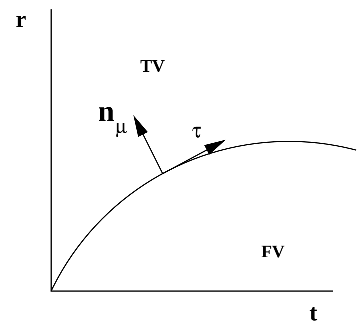
  - γ_{ab} - metric on the bubble wall worldsheet ($a, b = \tau, \theta, \phi$).
  - σ - surface energy density.
  - ϵ - difference in interior and exterior volume energies.
  - $g_{\mu\nu}$ - exterior space time metric ($\mu, \nu = t, r, \theta, \phi$).
- Exterior space time is Schwarzschild-de Sitter.

$$ds^2 = -a dt^2 + a^{-1} dr^2 + r^2 d\Omega^2,$$

$$a = 1 - \frac{2M}{r} - \frac{\Lambda_+}{3} r^2.$$

Unit Normals

- n_μ - the unit normal to the bubble wall world-sheet.



- Find from orthogonality ($g_{\mu\nu} n^\nu \partial_a x^\mu = 0$) and fixing the norm to be 1 ($g_{\mu\nu} n^\mu n^\nu = 1$).

$$n_t = -\frac{dr}{d\tau} = -r'$$

$$n_r = \frac{\sqrt{a + r'^2}}{a}$$

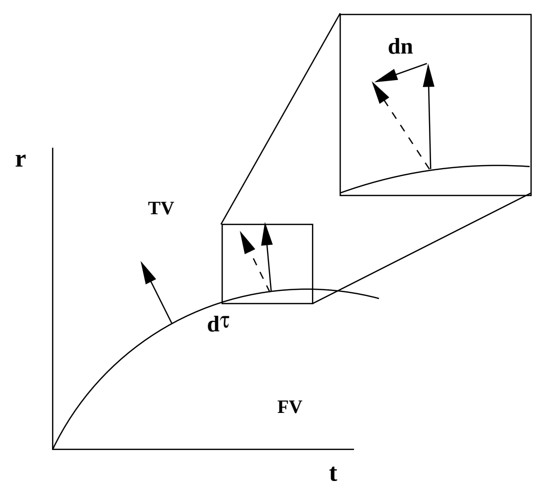
Simplified Wall Equation of Motion

- The equation of motion for the wall is (Guven):

$$\gamma^{ab} K_{ab} = -\frac{\epsilon}{\sigma}$$

  - K_{ab} - extrinsic curvature tensor.

$$K_{ab} = -\partial_a x^\mu \partial_b x^\nu D_\nu n_\mu ,$$



Wall Equation of Motion in SdS

- Plugging in the unit normals, the components of the extrinsic curvature tensor are

$$K_{\tau\tau} = \left[r'' + \frac{1}{2}\frac{da}{dr} \right] (a + r'^2)^{-1/2}$$

$$K_{\phi\phi} = -rat' \sin^2\theta = K_{\theta\theta} \sin^2\theta$$

- The equation of motion for $r(\tau)$ is then

$$r'' = \frac{\epsilon}{\sigma}\sqrt{a + r'^2} - \frac{2}{r}(a + r'^2) - \frac{1}{2}\frac{da}{dr}$$

Comparison to Junction Condition Evolution

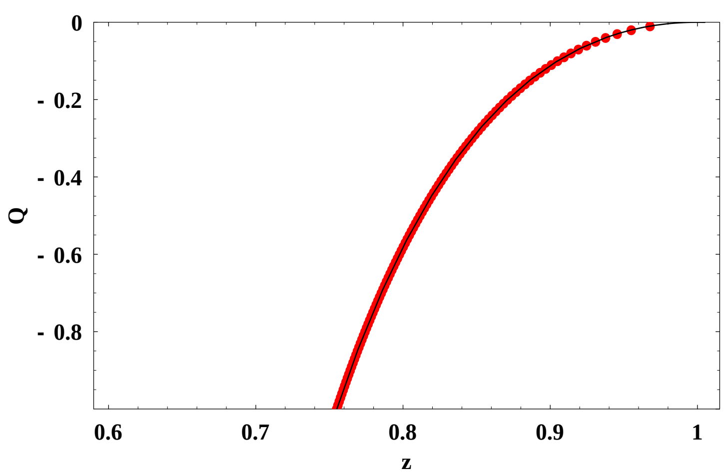
- Change variables ($A = \Lambda_+/k^2$, $B = \Lambda_-/k^2$, and $c = \left[|(A + B + 3)^2 - 4AB| \right]^{\frac{1}{2}}$):

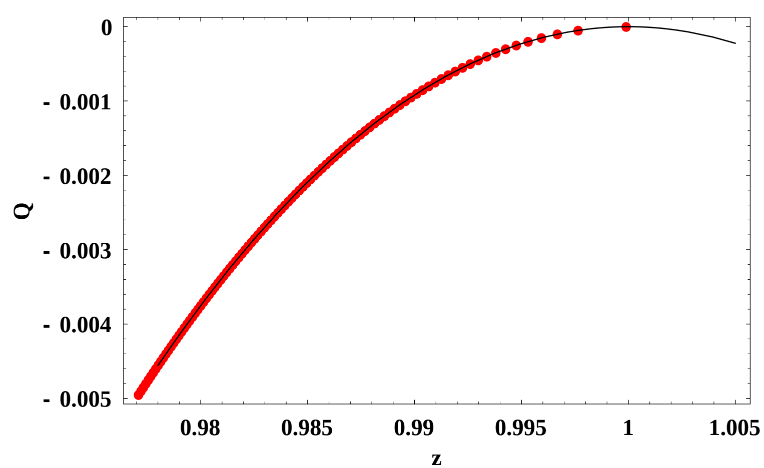
$$z'' = -\frac{3(B - A)}{c}\sqrt{a(-Q) + z'^2} - \frac{2}{z}(a(-Q) + z'^2) - \frac{(-Q)}{2}\frac{da}{dz}$$

- Use junction condition potential to set the initial velocity

$$z' = (Q - V(z_0))^{1/2}.$$

- Numerically integrate, and compare the position of the turning point (red dots) to junction condition potential (solid line). **!!Excellent Agreement!!**





Perturbations

- Physical perturbation: displacement normal to the wall.
- Set up a field $\phi(\xi)$ on the bubble.

$$\bar{x}^\mu = x^\mu + \phi(\xi) n^\mu; \qquad \phi(\xi) n^\mu << x^\mu$$

- Substitute $\bar{x}^\mu$ for x^μ in the action and expand to 2nd order in $\phi(\xi)$. The 1st order EOM for $\phi(\xi)$ is then (Garriga & Vilenkin and Guven):

$$\triangle\phi - \left[-R_{\mu\nu} h^{\mu\nu} + R^{(3)} - \frac{\epsilon^2}{\sigma^2} \right] \phi = 0$$

  - $\triangle\phi = \frac{1}{\sqrt{-\gamma}} \partial_a \left(\sqrt{-\gamma} \gamma^{ab} \partial_b \phi \right)$
  - $R_{\mu\nu}$ - Ricci tensor in background.
  - $h^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu$
  - $R^{(3)}$ - Ricci scalar on the bubble wall world sheet.

Perturbation Equation of Motion

- Decompose $\phi(\xi)$ into spherical harmonics $\phi(\xi) = \sum_{l,m} \phi_{lm}(\tau) Y_{lm}(\theta, \phi)$.
- Separate variables to get an equation for $\Phi_{lm}(\tau')$ (defined like z):

$$\Phi''_{lm} = -\frac{2z'}{z}\Phi'_{lm}$$

$$+\Big(\frac{108A}{c^2} + \frac{9(A-B)^2}{c^2} + \frac{12(B-A)}{cz}(a(-Q) + z'^2)^{1/2} + \frac{2(-Q)}{z^2}(4a-1)$$

$$+\frac{6z'^2}{z^2} + \frac{2(-Q)}{z}\frac{da}{dz} - \frac{l(l+1)(-Q)}{z^2} \Big) \Phi_{lm}$$

- If positive terms win out $\Longrightarrow$ perturbations grow.
- If negative terms win out, the solution is oscillatory.

Movie

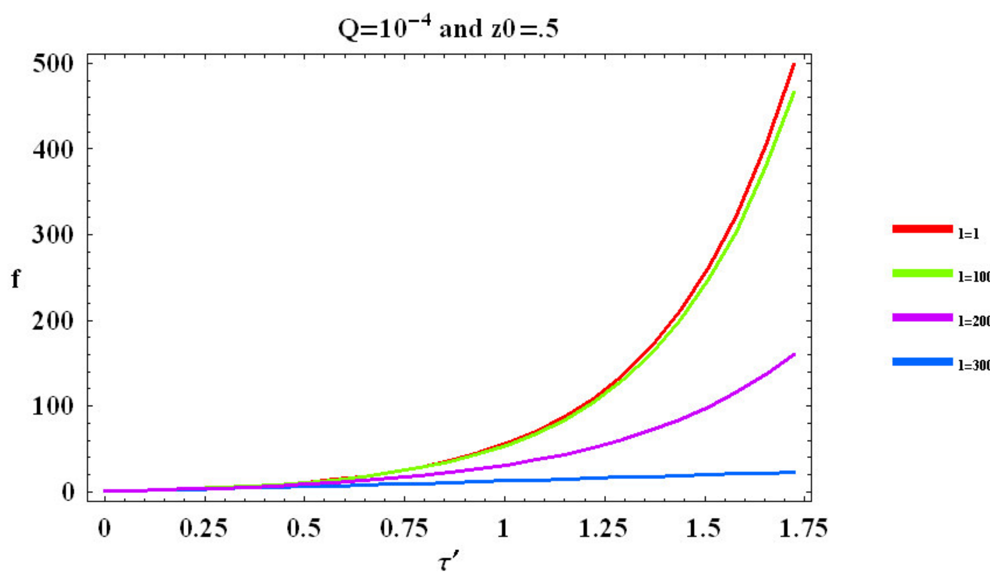
Evolution of modes $l = 1, 2, 3$ from zero to 1.

Solution

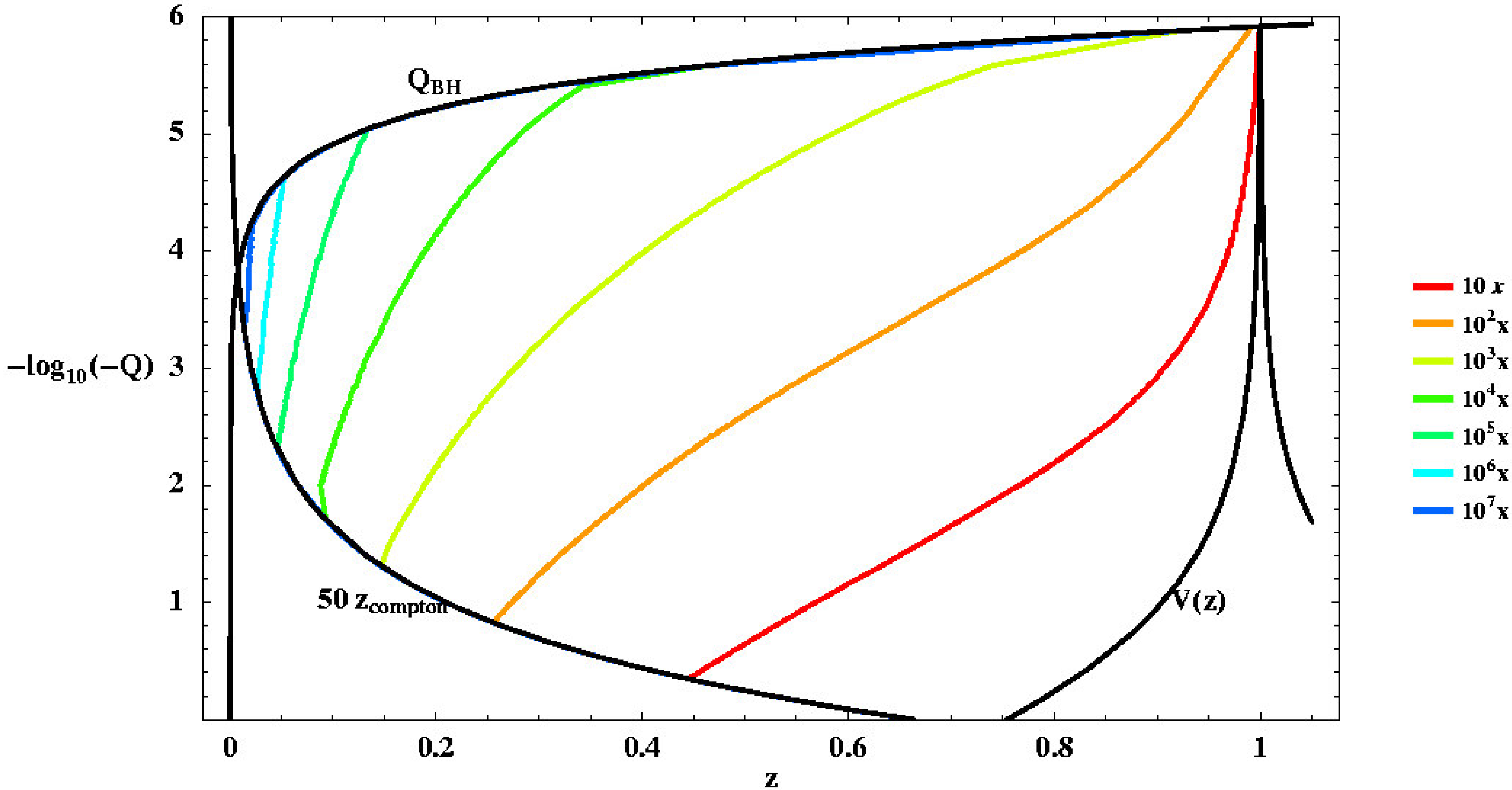
- Power series solution factors into:

$$\Phi_{lm}(\tau') = \Phi_{lm}(\tau' = 0) f(l, z_0, Q, \tau') + \dot{\Phi}_{lm}(\tau' = 0) g(l, z_0, Q, \tau')$$

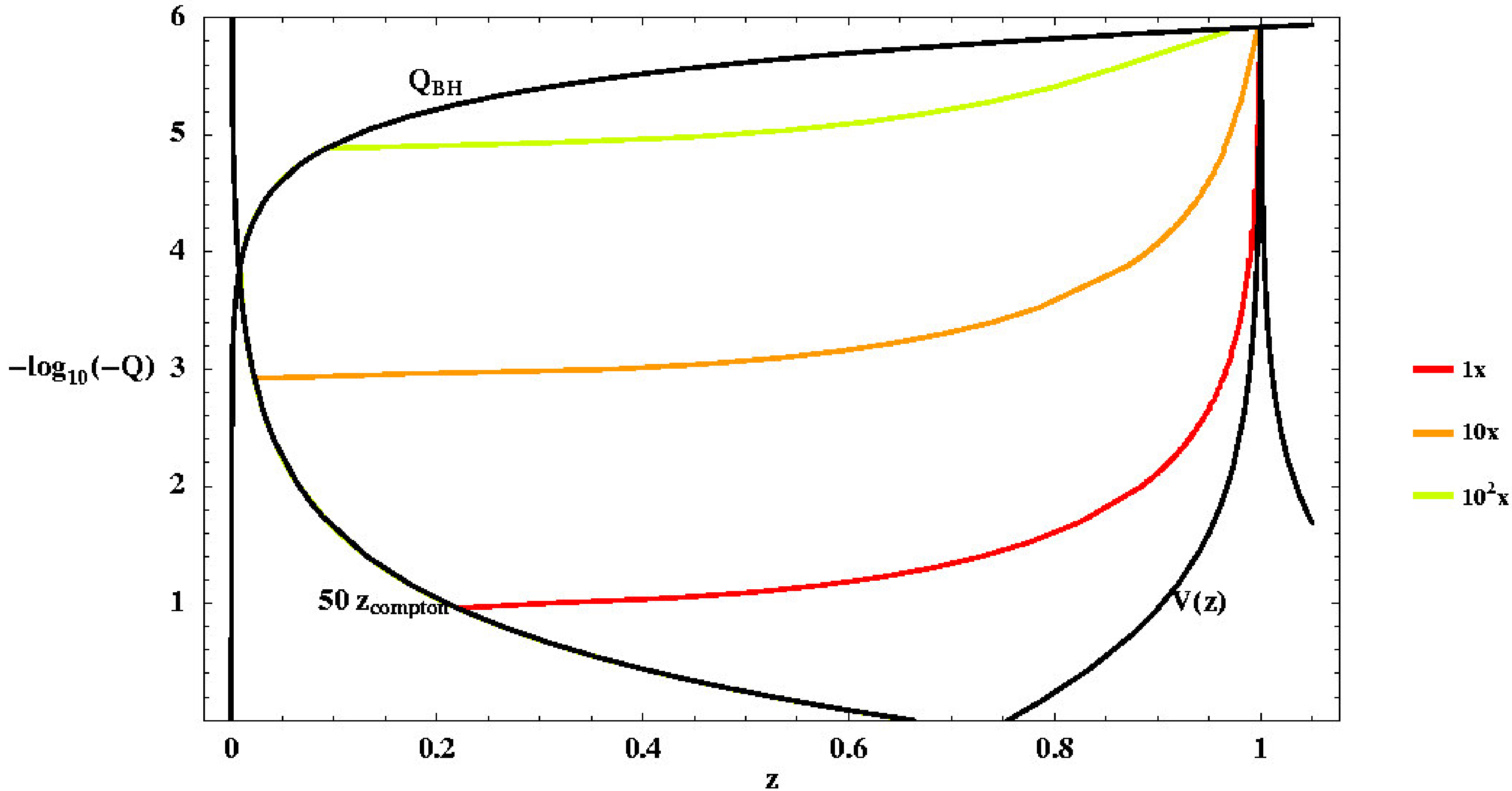
  - Solve numerically for $f(l, z_0, Q, \tau'_{max})$ and $g(l, z_0, Q, \tau'_{max})$ at the turning point.
- Which l-modes are unstable depends on what Q you are at.
  - Rough guide - when $l(l+1)(-Q) \leq 1$, solutions are always unstable.
  - $\Longrightarrow$ for small Q, many modes are unstable!



$f(z_0, Q, \tau'_{max})$ *for* $l = 1$ *and* $M_I = 10^{14}$ *GeV*



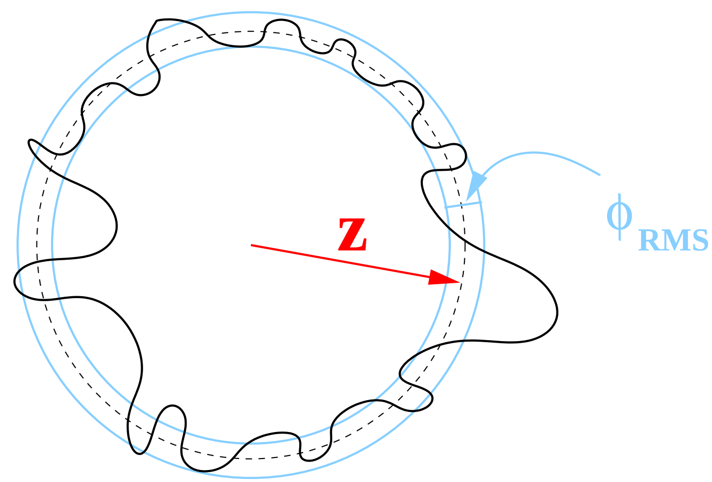
$g(z_0, Q, \tau'_{max})$ *for* $l = 1$ *and* $M_I = 10^{14}$ *GeV*



Non-Linear Evolution

- For linear analysis, need $\Phi(\tau,\theta,\phi) < z(\tau)$ for ALL τ, θ, and ϕ.
- Average over angles, condition for linearity is

$$\sqrt{\langle\Phi^2\rangle_{\theta,\phi}} = \left[\sum_l \frac{2l+1}{4\pi}\Phi_l^2(\tau)\right]^{1/2} < z(\tau)$$



Initial Conditions

- We still need to specify $\Phi_l(\tau' = 0)$ and $\dot{\Phi}_l(\tau' = 0)$.
- Best-Case scenario:
  - Bubbles are on quantum length scales: Assume minimum uncertainty.

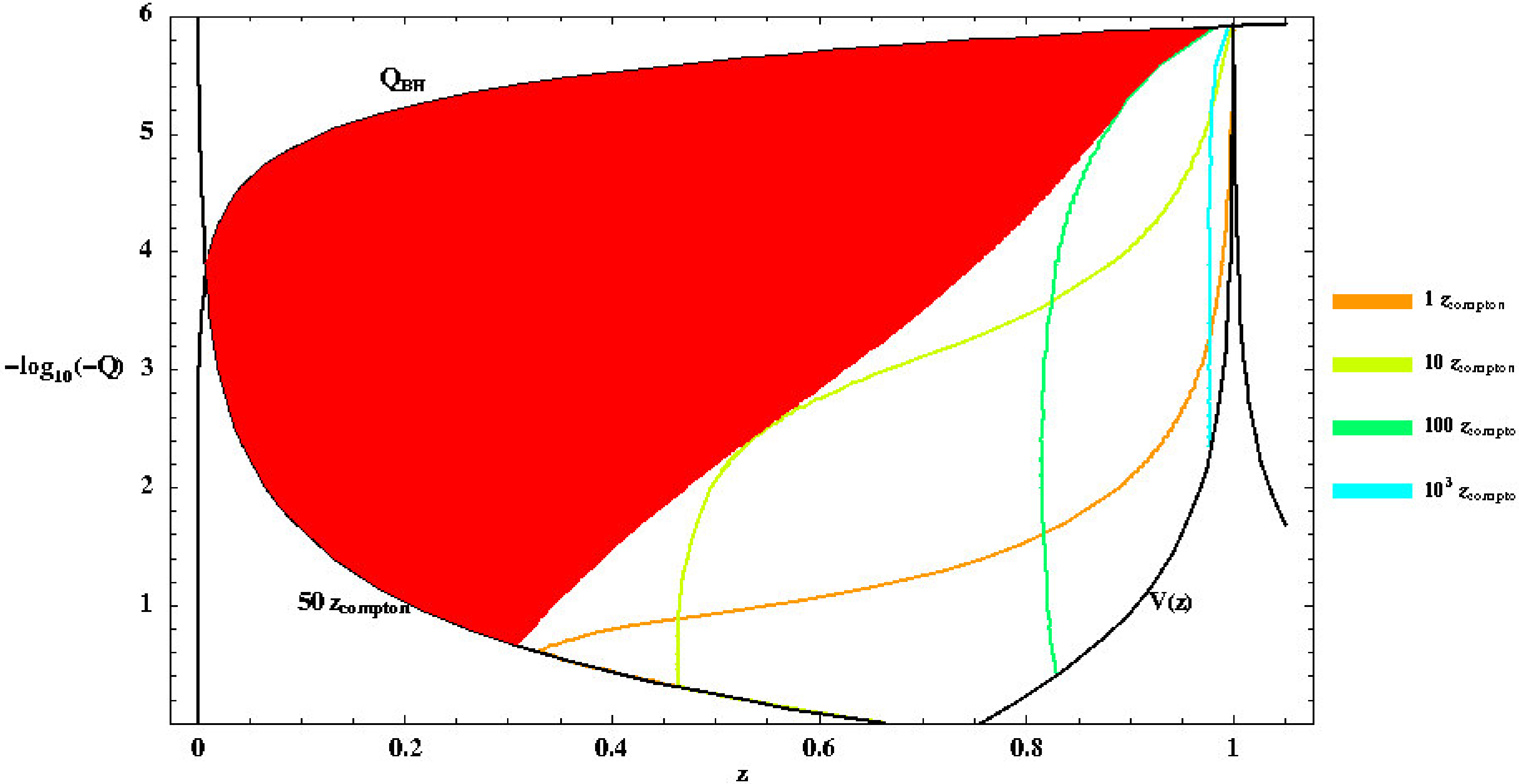
$$\Longrightarrow \dot{\Phi}(\tau' = 0) = [\Phi(\tau' = 0)]^{-1}$$

  - For each l-mode, minimize the perturbations under the above assumption.

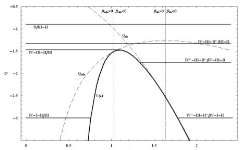
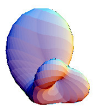
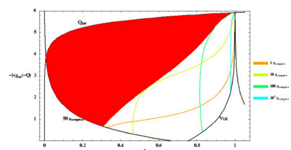
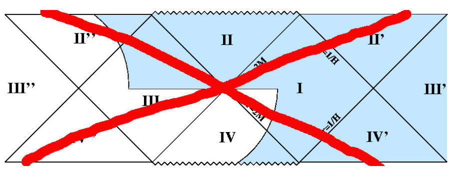
$$\Longrightarrow [\Phi_l(\tau'_{max})]_{min} = 2z^{1/2}_{compton}\sqrt{f(l, z_0, Q, \tau'_{max})g(l, z_0, Q, \tau'_{max})}$$

  - Re-sum the l modes averaged over all angles.

Non-linear Regions for $l = 1$ and $M_I = 10^{14}$ GeV



Conclusions

-  Cataloged all thin-walled bubble solutions.
-  Spherically symmetric bound solutions are unstable.
-  Some solutions go non-linear before the turning point.
- 

No tunneling from bound to unbound solutions?

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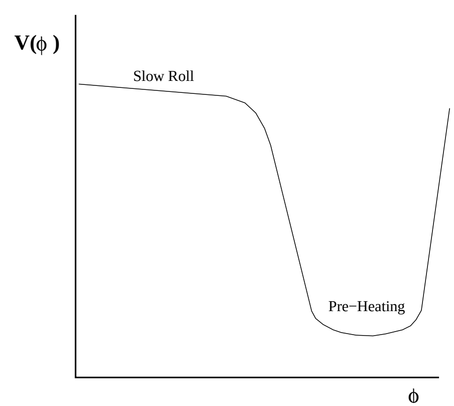
The End

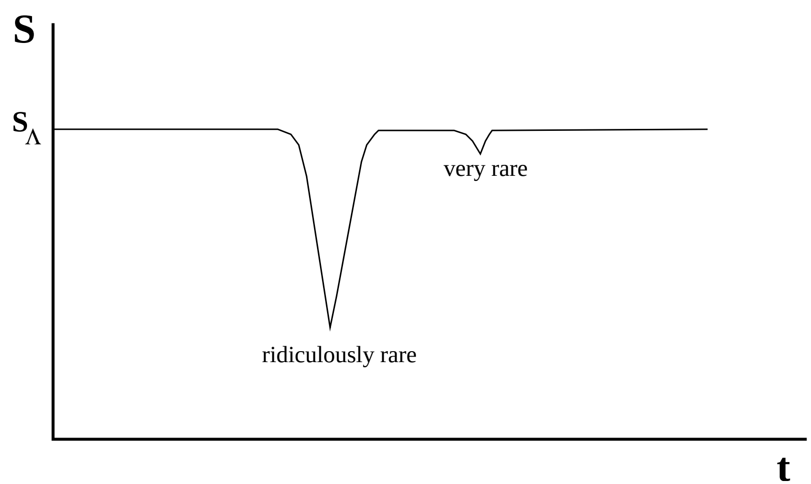
Inflation: A good fix.

- A scalar field ϕ (the inflaton) is postulated whose energy density generated a large effective cosmological constant.
  - The ϕ field had to acquire a large energy density in some spatial region.
  - This only happens if the potential energy is much greater than the kinetic energy: slow roll.
  - A period of inflation produces a universe which is homogenous, isotropic, flat, and devoid of monopoles by stretching.
- Inflation ends when the potential becomes steep.
  - The field oscillates around the minimum.
  - The energy in the inflaton is released as it decays: pre-heating.
  - The standard Big Bang cosmology picks up here.



de Sitter space as a meta-universe

- A universe with a cosmological constant Λ (like ours!) evolves to empty de Sitter space.
- de Sitter space has thermal characteristics much like a black hole.
- Fluctuations can occur which displace the system from equilibrium temporarily.
- Assume the only thing in the universe is a scalar field.
- Fluctuations in a scalar field can create topological defects, black holes, and baby universes.



Parameters

- Need to specify the surface energy density ($k = 4\pi\sigma$).
  - In our units ($\hbar = c = G = 1$) this is

$$k = 4\pi \frac{{M_I}^3}{{M_{pl}}^3}$$

- Need to specify the interior and exterior energy densities
  - In our units,

$$\Lambda_+ = \frac{{M_\Lambda}^4}{{M_{pl}}^4}; \quad A = \frac{{M_\Lambda}^4 {M_{pl}}^2}{(4\pi)^2 {M_I}^6}$$

$$\Lambda_- = \frac{{M_I}^4}{{M_{pl}}^4}; \quad B = \frac{{M_{pl}}^2}{(4\pi)^2 {M_I}^2}$$

A Quantum Cut

- The Compton wavelength of a piece of wall: lower bound on size.
  - Mass of a piece of wall

$$M = \pi s^2 \sigma$$

  - Compton wavelength is $s = M^{-1}$

$$\Longrightarrow s \simeq \sigma^{-1/3}$$

- Can't trust the evolution on scales less than $\simeq 50\sigma^{-1/3}$ as an arbitrary cut.

Penrose Theorem

- 1) Are there anti-trapped surfaces in the spacetime?
  - Incoming and outgoing (through a closed surface) null rays are both diverging.
- 2) Does the null energy condition hold?

$$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0 \text{ for all null } k^{\mu}$$

If yes to both $\Longrightarrow$ an initial singularity is inevitable

The exception: