Last class was very qualitative.
We'll have more to say soon about how string theory fits into the big picture.

Zwiebach Chap. 2, Secs. 2.1–2.6

Today, we do some work:
- review special relavity
- introduce lightcone coordinates

String theory is QM of a relativistic vibrating string.

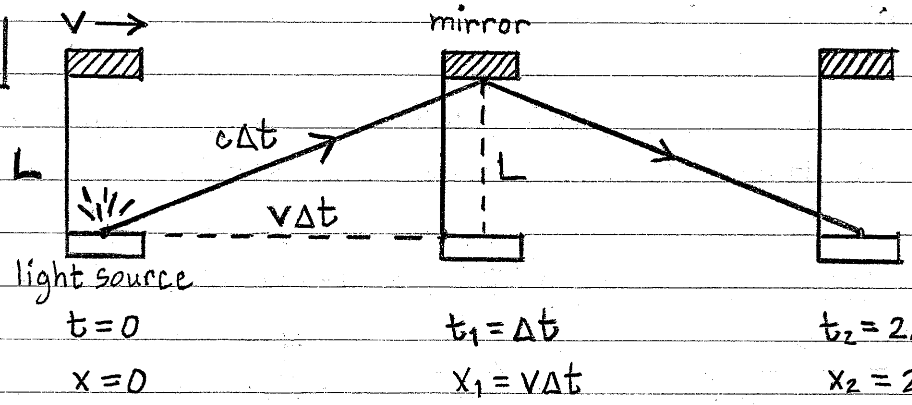
Special Relativity

Gedankenexperiments

1. Time dilation (moving clocks run slow)
2. Lorentz contraction (length contraction parallel to motion)

Gedankenexperiment 1:

Frame S

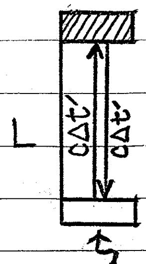


$$L = \sqrt{(c\Delta t)^2 - (v\Delta t)^2} = c\Delta t\sqrt{1-\frac{v^2}{c^2}} = c\Delta t/\gamma,$$

where $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}} > 1$ (Lorentz factor).

Now consider the same experiment in the rest frame S' of the apparatus.

Frame S'



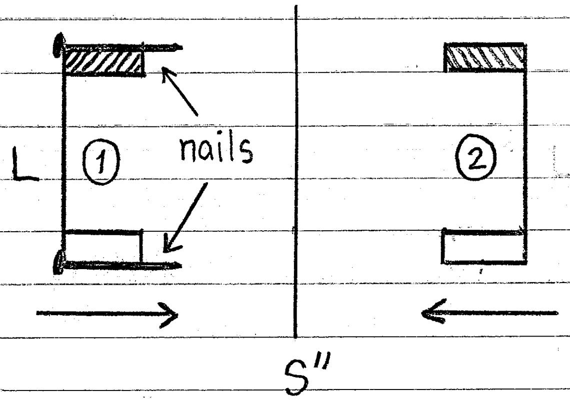
(at rest)

$\Delta t' = L/c$. So, using previous result,

$$\boxed{\Delta t' = \Delta t/\gamma}$$ time dilation

(moving clocks run slow).

[Why is L the same for S and S'?



Consider 2 copies of apparatus moving in symmetric frame.

Nails of ① just scratch ends of ② in this frame (by symmetry) so transverse lengths are the same in all frames.]

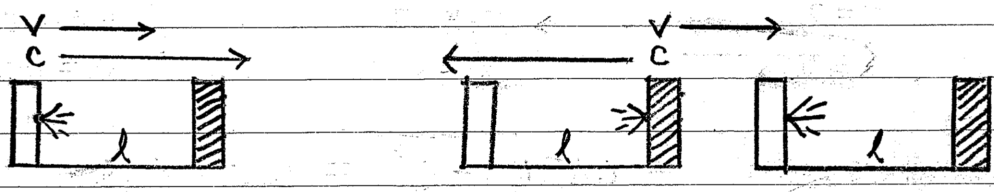
Experimental tests?

dilation of muon lifetimes in atmosphere ($\mu^- \to e^- + \bar{\nu}_e + \nu$)

1971 Hafel + Keating, cesium clock 40 hrs on airplanes

Gedankenexperiment 2: rotate apparatus 90° and repeat experiment.

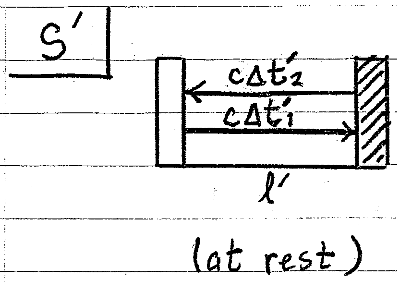
S



$t=0$ $\quad t=\Delta t_1$ $\quad t=\Delta t_1+\Delta t_2$

$\Delta t_1 = l/(c-v)$ $\quad \Delta t_2 = l/(c+v)$ (return trip is shorter)

$$\Delta t = \Delta t_1 + \Delta t_2 = l\left(\frac{1}{c-v} + \frac{1}{c+v}\right) = \frac{2lc}{c^2-v^2} = \frac{2l}{c}\gamma^2.$$



$$\Delta t' = \Delta t_1' + \Delta t_2' = \frac{2\ell'}{c}$$

Also, from time dilation result,

$$\Delta t/\gamma = \Delta t' \implies \frac{2\ell}{c}\gamma = \frac{2\ell'}{c}.$$

That is, $\boxed{\ell = \ell'/\gamma}$ Lorentz contraction

(moving stick is shortened parallel to motion).

In Problem Set 1, will show that

Gedankenexperiments 1+2 $\Rightarrow$ Lorentz transformations for boosts:

$$ct' = \gamma(ct - \beta x), \qquad x' = \gamma(x - \beta ct),$$

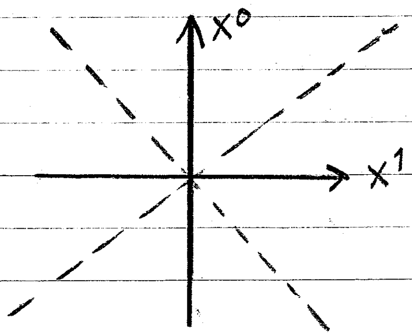
or

$$x'^0 = \gamma(x^0 - \beta x^1), \qquad x'^1 = \gamma(x^1 - \beta x^0).$$

Here, $\beta = v/c$ and $(x^0, x^1) = (ct, x)$.

Let's get some intuition on Lorentz transformations and their implications for simultaneity (same time) and coincidence (same spacial location) through

Spacetime diagrams.

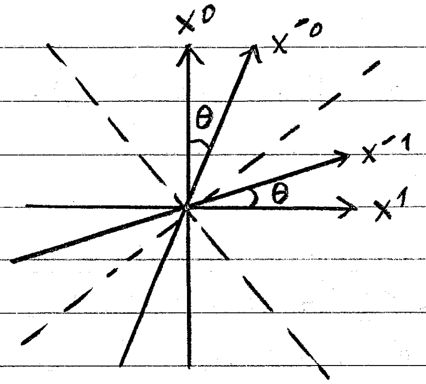


- Vertical axis is time x^0
- Horozontal axis is space x^1
- 45° dotted lines are light cone (by analogy to higher dim case where light rays through origin lie on cone $(x^0)^2 - [(x^1)^2 + (x^2)^2 + (x^3)^2] = 0$).

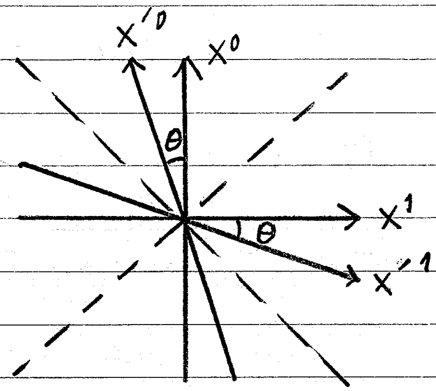
Zwiebach Problem 2.5: What do x'^0, x'^1 axes look like on x^0, x^1 spacetime diagram?

x'^1 axis: $x'^0 = 0 \Rightarrow x^0 = \beta x^1$, slope β.
x'^0 axis: $x'^1 = 0 \Rightarrow x^0 = \beta^{-1} x^1$, slop $1/\beta$.

(lightcone is invariant: $x'^0 = x'^1 \Rightarrow x^0 = x^1$
($ct' = x' \Rightarrow ct = x$).)



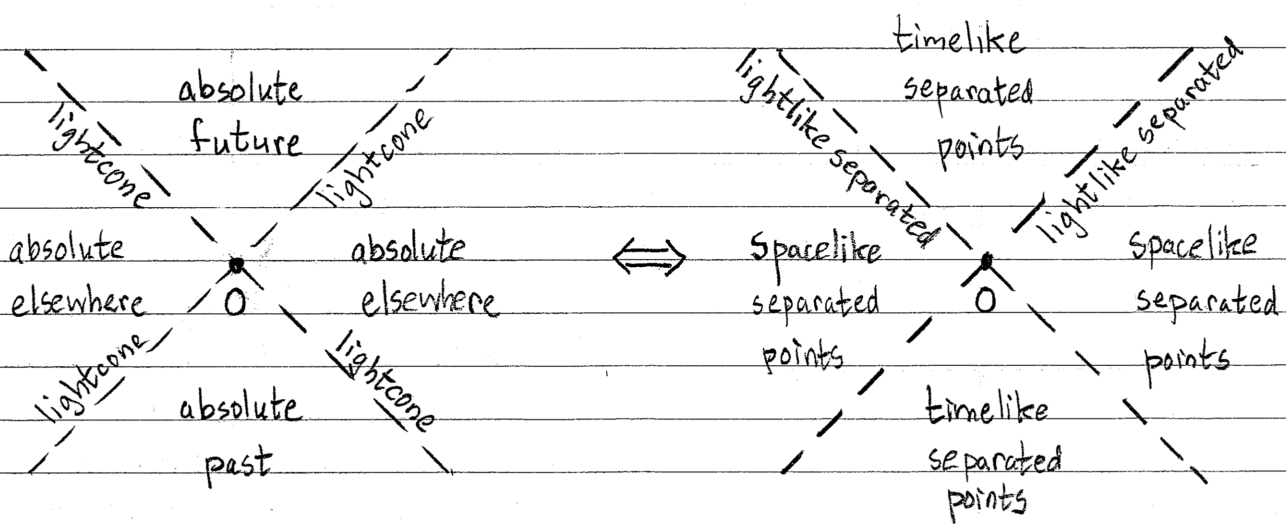
[x'^0, x'^1 axes approach lightcone as $\beta \to \pm 1$ ($v \to \pm c$).]



$\theta = \tan^{-1}\beta$, $\beta > 0$.

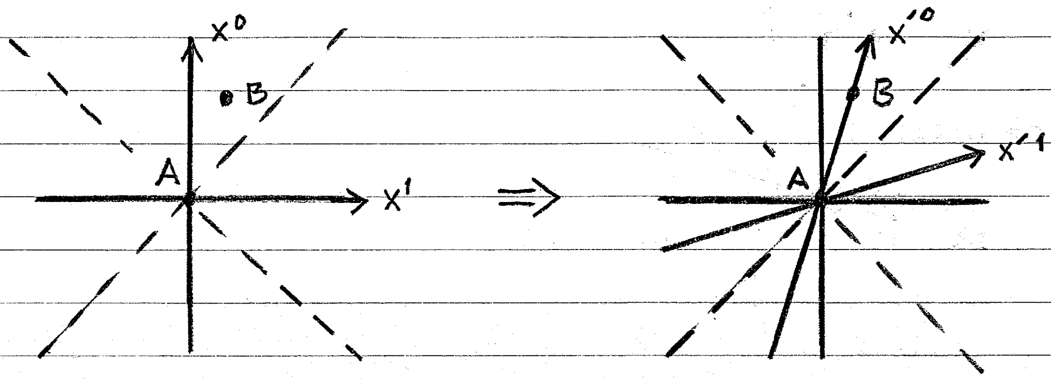
$\theta = \tan^{-1}(-\beta)$, $\beta < 0$.

Lightcone divides spacetime into points that are in the absolute future (of the origin O), the absolute past, and absolute elsewhere. The boundary is the lightcone.



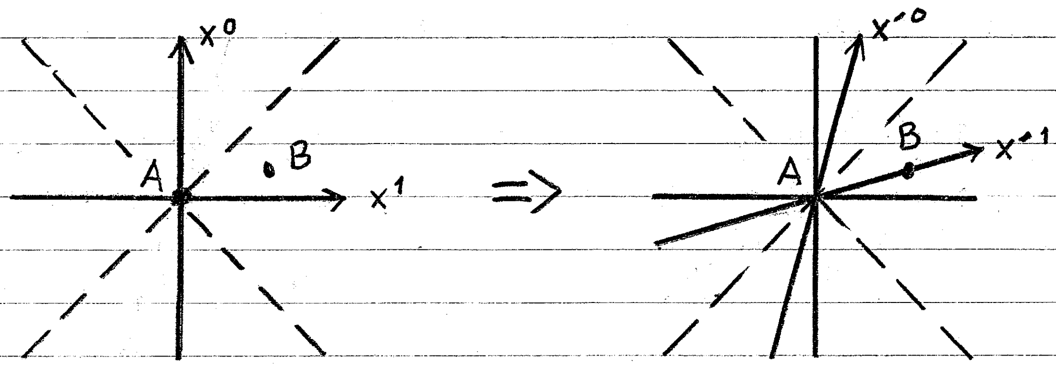
The second figure will be clear when we discuss the spacetime interval Δs^2. To understand the first figure:

- Suppose that event B is in the absolute future of event A in some frame S:



Then there exists a frame S' with A, B coincident (same x'^1).
There is no frame with A, B simultaneous (B always in future of A).

- Suppose that event B is in the absolute elsewhere of event A in some frame S.



Then there exists a frame S' with A, B simultaneous (same x'^0).
There is no frame with A, B coincident (B always elsewhere from A).

Define spacetime interval (btw. event A at x and B at $x+\Delta x$):

$$-\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2.$$

Then, for $\Delta s^2 > 0$ we call Δx timelike,
$= 0$ lightlike,
< 0 spacelike.

(Cf. p. 4 of notes.)

Define also the Minkowski metric:

$$\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}.$$

Then the infinitesimal spacetime interval is

$$-ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu.$$ Einstein summation convention: sum over matching pairs of upper + lower indices.

(In higher dimensions, just add $(dx^5)^2 + (dx^6)^2 + \cdots$.)

What linear transformations leave $-ds^2$ invariant?

$$x'^\mu = L^\mu{}_\nu x^\nu + a^\mu$$
$$dx'^\mu = L^\mu{}_\nu\, dx^\mu$$

$$-ds'^2 = \eta_{\rho\sigma}\, dx'^\rho dx'^\sigma = \eta_{\rho\sigma} L^\rho{}_\mu dx^\mu L^\sigma{}_\nu dx^\nu$$
$$= (\eta_{\rho\sigma} L^\rho{}_\mu L^\sigma{}_\nu)\, dx^\mu dx^\nu$$
$$= -ds^2, \text{ if } \boxed{\eta_{\rho\sigma} L^\rho{}_\mu L^\sigma{}_\nu = \eta_{\mu\nu}.}$$

This equation defines the group of Lorentz transformations.
If we also include the translations a^μ, we obtain the Poincaré group.

Can also write $L^{\rho}{}_{\mu}\,\eta_{\rho\sigma}\,L^{\sigma}{}_{\nu} = \eta_{\mu\nu}$

$$(L^T)_{\mu}{}^{\rho}\,\eta_{\rho\sigma}\,L^{\sigma}{}_{\nu} = \eta_{\mu\nu}$$

$$\Longleftrightarrow \quad L^T \eta L = \eta, \quad \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

<u>Lorentz transformations generalize spacial rotations</u>

$$\Delta x'^0 = \Delta x^0, \quad (\Delta x'^1)^2 + (\Delta x'^2)^2 + (\Delta x'^3)^2 = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$

$$(\uparrow \text{length}^2 = \text{rotational invariant}),$$

by combining rotations with boosts.

<u>Basic operations involving 4-vectors:</u>

4-vectors a^μ are quantities that transform like x^μ under Lorentz transformations (for example, $p^\mu = (E/c, \vec{p})$).

Given two 4-vectors a^μ, b^μ and $\eta_{\mu\nu}$, we define:

scalar product $\quad a\cdot b = \eta_{\mu\nu} a^\mu b^\nu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3,$

lowering $\quad a_\mu = \eta_{\mu\nu} a^\nu = (-a^0, a^1, a^2, a^3),$

inverse metric $\quad \eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1),$

$$\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu = \begin{cases} 1 & \mu=\nu \\ 0 & \mu\neq\nu \end{cases} \quad (\text{Kronecker } \delta),$$

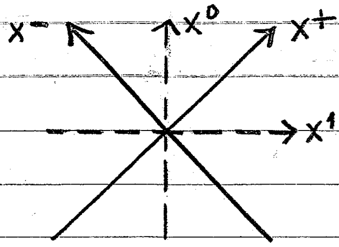
$$a\cdot b = \eta^{\mu\nu} a_\mu b_\mu,$$

raising $\quad a^\mu = \eta^{\mu\nu} a_\nu.$

Quantization of the relativistic string is subtle.
As we will see, it involves "gauge fixing" of unphysical dof.
In this course, we will use lightcone gauge.
So, need to understand:

Lightcone coordinates.

(This involves an arbitrary choice.)



Define $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^1)$ LC time,
$x^- = \frac{1}{\sqrt{2}}(x^0 - x^1)$ LC space,

$$-2dx^+dx^- = -(dx^0)^2 + (dx^1)^2.$$

Spacetime interval is: $-ds^2 = -2dx^+dx^- + (dx^2)^2 + (dx^3)^2$
$= \hat{\eta}_{\mu\nu}\, dx^\mu dx^\nu \quad (\mu,\nu = +,-,2,3)$,

where

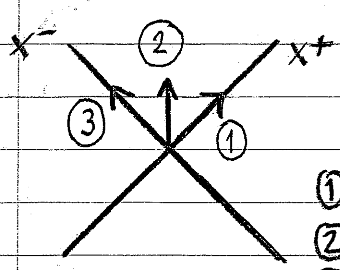
$$\hat{\eta}_{\mu\nu} = \left(\begin{array}{cc|cc} & -1 & & \\ -1 & & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right),$$

symmetric since $a\cdot b = b\cdot a$.

So, $x_- = -x^+$ and $x_+ = -x^-$.

Let's get some intuition for lightcone coordinates by considering constant velocity motion $x^1 = \beta x^0$ in LC frame.

$$\left.\begin{array}{l} x^+ = \frac{1}{\sqrt{2}}(1+\beta)x^0 \\ x^- = \frac{1}{\sqrt{2}}(1-\beta)x^0 \end{array}\right. \Rightarrow \quad x^- = \left(\frac{1-\beta}{1+\beta}\right)x^+.$$



LC velocity: $v_{LC} = \dfrac{dx^-}{dx^+} = \dfrac{1-\beta}{1+\beta} > 0.$

1: $v_{LC} \to 0$ as $v \to c$ and remain at fixed spatial x^- for all LC time x^+.
2: $v_{LC} \to 1$ as $v \to 0$ and move along $x^+ = x^-$.
3: $v_{LC} \to \infty$ as $v \to -c$ and move along x^- axis at fixed LC time x^+.

Lightcone energy and momentum.

$$p^+ = \frac{1}{\sqrt{2}}(p^0 + p^1),$$
$$p^- = \frac{1}{\sqrt{2}}(p^0 - p^1).$$

Lightcone energy? Both p^+ and $p^- > 0$. Zwiebach decribes a simple way to determine which should be LC energy:

In QM, energy × time appears in exponent of plane wave.

$$\psi = \exp\left(\frac{i}{\hbar} p \cdot x\right) = \exp\left(\frac{i}{\hbar}(-\underset{\substack{\uparrow \\ E = p^0 c}}{Et} + \vec{p} \cdot \vec{x})\right),$$

$$i\hbar c \frac{\partial \psi}{\partial (ct)} = E\psi.$$

So, in lightcone coordinates,

$$\psi = \exp\left(\frac{i}{\hbar}(-p^- x^+ - p^+ x^- + p^2 x^2 + p^3 x^3)\right),$$

$$i\hbar c \frac{\partial \psi}{\partial x^+} = E_{LC}\psi$$

$$\Rightarrow \quad \boxed{E_{LC} = p^- c.}$$

For motion in the positive x^1 direction,

$$p^0 = \frac{E}{c} = \sqrt{(p^1)^2 + m^2c^2} = p^1\sqrt{1 + \frac{m^2c^2}{(p^1)^2}}$$
$$\simeq p^1 + \frac{m^2c^2}{2p^1}$$

$$\Rightarrow p^- = \frac{1}{\sqrt{2}}(p^0 - p^1) \simeq \frac{m^2c^2}{2\sqrt{2}\,p^1}.$$

E_{LC} decreases as p^1 increases, just as v_{LC} decreases as β increases.