Sufficient condition on noise correlations for scalable quantum computing

John Preskill, 2 February 2012

Is quantum computing scalable? The accuracy threshold theorem for quantum computation establishes that scalability is achievable if the noise afflicting the quantum computer is not too strong and not too strongly correlated. For scalability to fail as a matter of principle then, either the accepted principles of quantum physics must fail for complex highly entangled systems (as 't Hooft [tH99] has suggested), or else either the Hamiltonian or the quantum state of the world must impose noise correlations that overwhelm fault-tolerant quantum protocols (as Kalai [K11] has suggested). To get further insight into this issue, it is useful to derive conditions on the Hamiltonian that suffice for scalability.

The noise model we consider is formulated by specifying a time-dependent Hamiltonian H that governs the joint evolution of the system and the bath, which can be expressed as

$$H = H_S + H_B + H_{SB}; \tag{1}$$

here H_S is the time-dependent Hamiltonian of the system that realizes an ideal quantum circuit, H_B is the Hamiltonian of the bath, and H_{SB}, which describes the coupling of the system to the bath, is the origin of the noise. We place no restrictions on the bath Hamiltonian H_B. Without any loss of generality, we may expand the system-bath Hamiltonian in the form

$$H_{SB} = \sum_{k=1}^{\infty} \sum_{\langle i_i, i_2, \ldots i_k \rangle} H^{(k)}_{i_i, i_2, \ldots i_k} = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_i, i_2, \ldots i_k} H^{(k)}_{i_i, i_2, \ldots i_k}. \tag{2}$$

Here, $H^{(k)}_{i_i, i_2, \ldots i_k}$ acts on the k qubits labeled by the indices $i_i, i_2, \ldots i_k$, and also acts on the bath; for each k we sum over all ways of choosing k system qubits. We use $\langle i_i, i_2, \ldots i_k \rangle$ to denote an unordered set of k qubits; by definition, $H^{(k)}_{i_i, i_2, \ldots i_k}$ is invariant under permutations of the k qubits and vanishes if two of the indices coincide. Hence the two expressions for H_{SB} are equivalent. We will not need to assume anything about the initial state of the bath, except that the system qubits can be well enough isolated from the bath that we can prepare single-qubit states with reasonable fidelity.

We use the term *location* to speak of an operation in a quantum circuit performed in a single time step; a location may be a single-qubit or multi-qubit gate, a qubit preparation step, a qubit measurement, or the identity operation in the case where a qubit is idle during a time step. We model a noisy preparation as an ideal preparation followed by evolution governed by H, and a noisy measurement as an ideal measurement preceded by evolution governed by H. It is convenient to imagine that all system qubits are prepared at the very beginning of the computation and measured at the very end; in that case the noisy computation can be fully characterized by a unitary evolution operator U acting jointly on the system and the bath, obtained by solving the time-dependent Schrödinger equation for the Hamiltonian Eq. (2).

Our goal is to derive from Eq. (2) an expression for the *effective noise strength* ε of the noisy computation, which is defined as follows. We envision performing a formal expansion of U in powers of the perturbation H_{SB}, to all orders. Consider a particular set $\mathcal{I}_r$ of r circuit locations, and let $E(\mathcal{I}_r)$ denote the sum of all terms in the expansion such that every location in $\mathcal{I}_r$ is faulty, *i.e.*, such that at least one of the qubits at that location is struck at least once by a term in H_{SB} during the execution of the gate. We say that the noise has effective noise strength ε if

$$\|E(\mathcal{I}_r)\| \leq \varepsilon^r \tag{3}$$

for any set $\mathcal{I}_r$. The accuracy threshold theorem for quantum computing shows that scalable quantum computing is possible if ε is less than a positive constant ε_0 [AGP2006, NP2009].

Let us define

$$\tilde{\eta}_1^{(k)} = \max_{i_1} \sum_{i_2, i_3, \ldots, i_k} \|H^{(k)}_{i_1, i_2, i_3, \ldots, i_k}\| \, t_0, \tag{4}$$

where the maximum is over all qubits and all times, and where t_0 is the maximal duration of any location. Then our main result an be stated as follows.

Theorem 1. (Effective noise strength for correlated Hamiltonian noise) *If each quantum gate acts on at most m qubits and if*

$$\tilde{\eta}_1^{(k)} \leq f_k \alpha^k, \tag{5}$$

for all k, then

$$\varepsilon \leq 2m\alpha \ \exp\left(\sum_{k=1}^{\infty} \frac{g_k}{2k!}\right), \tag{6}$$

where

$$g_k = f_k + \sum_{l=1}^{\infty} \frac{(k-1)! f_{k+l} (2\alpha)^l}{(k+l-1)!}. \tag{7}$$

It follows that quantum computing is scalable provided the strength of k-qubit interactions decays sufficiently rapidly with k (so that the sums in Eq. (6) and Eq. (7) converge), and also decays as the spatial separation of the qubits increases (so that the sum defining $\tilde{\eta}_1^{(k)}$ in Eq. (4) converges).

If, for example, $f_k = 1$, then

$$g_k \leq \sum_{l=0}^{\infty} (2\alpha)^l = (1-2\alpha)^{-1} \equiv C(\alpha), \tag{8}$$

and hence

$$\varepsilon \leq 2m\alpha \left(e^{(e-1)/2}\right)^{C(\alpha)} \approx 4.72 \ m\alpha, \tag{9}$$

where the last approximation uses $C(\alpha) \approx 1$ for $\alpha \ll 1$, as is the case if ε is smaller than the threshold value $\varepsilon_0 \approx 10^{-4}$. If instead

$$f_k \leq k!/k^p, \tag{10}$$

then

$$\begin{aligned}
g_k \quad &\leq \quad \frac{k!}{k^p}\left(\sum_{l=0}^{\infty} \frac{(k-1)! f_{k+l} (2\alpha)^l}{(k+l-1)! f_k}\right) = \frac{k!}{k^p}\left(\sum_{l=0}^{\infty} \frac{(k-1)!(k+l)! k^p (2\alpha)^l}{(k+l-1)! k! (k+l)^p}\right) \\
&\leq \quad \frac{k!}{k^p}\left(\sum_{l=0}^{\infty} \frac{(k+l)(2\alpha)^l}{k}\right) = \frac{k!}{k^p}\left(\sum_{l=0}^{\infty}\left(1+\frac{l}{k}\right)(2\alpha)^l\right) \\
&= \quad \frac{k!}{k^p}\left(\frac{1}{1-2\alpha} + \frac{1}{k}\left(\frac{2\alpha}{(1-2\alpha)^2}\right)\right) \leq \frac{k!}{k^p}\left(\frac{1}{1-2\alpha}\right)^2 = C(\alpha)^2 \frac{k!}{k^p}.
\end{aligned} \tag{11}$$

where we obtained the inequality in the last line using $k \geq 1$. For $p = 2$, for example, we find

$$\varepsilon \leq 2m\alpha \left(e^{\pi^2/12}\right)^{C(\alpha)^2} \approx 4.55 \ m\alpha, \tag{12}$$

again using $C(\alpha) \approx 1$ to obtain the numerical expression. For $p > 1$ the sum over k in Eq. (6) converges, and hence we obtain a finite expression for ε. Therefore, scalable fault-tolerant quantum computation is achievable for a sufficiently small (positive) value of α.

In [AKP06], scalability was proven for the special case in which only the $k = 2$ term in the Hamiltonian is nonzero. To prove Theorem 1 we generalize the ideas used in [AKP06]. We write the system-bath Hamiltonian as

$$H_{SB} = \sum_a H_{SB,a} \tag{13}$$

where a is a shorthand for the indices k, and $i_i, i_2, \ldots i_k$ in Eq. (2). For the sake of conceptual clarity we imagine dividing time into infinitesimal intervals, each of width Δ, and express the time evolution operator for the interval $(t, t+\Delta)$ as

$$U(t+\Delta, t) \approx e^{-i\Delta H} \approx e^{-i\Delta H_S} e^{-i\Delta H_B} \prod_a (I_{SB} - i\Delta H_{SB,a}). \tag{14}$$

(We have omitted terms higher order in $\|H\|\Delta$; strictly speaking, then, to justify Eq. (14) we should regulate the bath Hamiltonian by imposing an upper bound on its norm, then choose Δ small enough so these higher order terms can be safely neglected.) We expand $U(t+\Delta, t)$ as a sum of monomials, where for each value of a either I_{SB} or $-i\Delta H_{SB,a}$ appears; then we obtain the perturbation expansion of the full time evolution operator U over time T by stitching together T/Δ such infinitesimal time evolution operators.

We will refer to the r specified locations in the set $\mathcal{I}_r$ as the "marked locations" and to the remaining locations as the "unmarked locations." For now, suppose for definiteness that all of the marked locations are single-qubit gates. For any term in the perturbation expansion contributing to $E(\mathcal{I}_r)$ there must be an *earliest* infinitesimal time interval in each of the r marked locations where a term $H_{SB,a}$ acts nontrivially on that qubit. Suppose we fix the infinitesimal time intervals where these earliest insertions of H_{SB} occur, and also fix the terms $\{H_{SB,a}\}$ in the system-bath Hamiltonian that act there, but sum over all the terms in the perturbation expansion acting in other time intervals and on other qubits. Then in between the fixed earliest insertions in the marked locations, the joint evolution of the system and the bath is governed by a modified Hamiltonian

$$H^{(\text{modified})} = H_S + H_B + H_{SB}^{(\text{modified})}, \quad H_{SB}^{(\text{modified})} = \sum_a^{(\text{modified})} H_{SB,a}, \tag{15}$$

where the modified sum excludes any term $H_{SB,a}$ acting nontrivially on any one of the marked locations during any time interval prior to the fixed time of the earliest insertion. The important point is that the time evolution operator in between successive insertions of the perturbation is unitary and hence has unit operator norm. We conclude, then, that the contribution to $E(\mathcal{I}_r)$ with the earliest insertions at the marked locations fixed has operator norm bounded above by

$$\prod_a^{(\text{earliest})} \left(\|H_{SB,a}\|\Delta\right), \tag{16}$$

where the product is over the terms in the system-bath Hamiltonian that act at the earliest insertions. To bound $\|E(\mathcal{I}_r)\|$, we sum over the t_0/Δ time intervals at each location where the earliest insertion may occur, and also sum over all the ways of choosing the term $H_{SB,a}$ that acts at each insertion, obtaining

$$\|E(\mathcal{I}_r\| \le \sum_{\{H_{SB,a}\}}^{(\text{insertions})} \prod_a^{(\text{earliest})} \left(\|H_{SB,a}\|t_0\right). \tag{17}$$

Summing over the possible intervals for the first insertion turns the factor Δ into the factor t_0.

Now we have to figure out how to sum over all ways of choosing the terms $\{H_{SB,a}\}$ acting at the earliest insertions inside the r marked circuit locations. Since H_{SB} contains multi-qubit terms, a single term in H_{SB} can simultaneously produce the first insertion at multiple circuit locations occurring in the same time step.

Specifically, a single term in $H^{(k)}$ might cause simultaneous faults in j of the r marked locations for any $j \le k$. We use the term "j-contraction" to refer to the case when a single term in Eq. (2) produces the first insertion in each of j marked locations.

The strength of a one-contraction can be bounded by

$$\eta_1 = \sum_{l=0}^{\infty} \eta_1^{(1+l)}, \tag{18}$$

where

$$\eta_1^{(1+l)} = \max_{i_1}^{(\text{in})} \sum_{\langle j_1, j_2, \ldots, j_l \rangle}^{(\text{all})} \| H^{(1+l)}_{i_1, j_1, j_2, \ldots, j_l} \| \; t_0 = \max_{i_1}^{(\text{in})} \frac{1}{l!} \sum_{j_1, j_2, \ldots, j_l}^{(\text{all})} \| H^{(1+l)}_{i_1, j_1, j_2, \ldots, j_l} \| \; t_0 \tag{19}$$

Here i_1 is one of the marked locations, and other indices are summed over all locations (both marked and unmarked), to allow for the possibility that the insertion at a marked location need not be the *first* insertion; hence the higher order ($l > 0$) terms in Eq. (19) can be contributions to the strength of the one-contraction rather than a j-contraction for $j > 1$ even though some of the locations in $\{j_1, j_2, \ldots, j_l\}$ may be marked.

Similarly, for $k > 1$, the strength of a k-contraction can be bounded by

$$\eta_k = \sum_{l=0}^{\infty} \eta_k^{(k+l)}, \tag{20}$$

where

$$\eta_k^{(k+l)} = \sum_{\langle i_1, i_2, \ldots, i_k \rangle}^{(\text{in})} \sum_{\langle j_1, j_2, \ldots, j_l \rangle}^{(\text{all})} \| H^{(k+l)}_{i_1, i_2, \ldots i_k, j_1, j_2, \ldots, j_l} \| \; t_0. \tag{21}$$

Here for the "in" sum the qubits are restricted to the marked locations and for the "all" sum they may be at either marked or unmarked locations.

By summing all ways of choosing the first insertion in each of r marked locations, we obtain the bound

$$\varepsilon^r \le \sum_{r_1, r_2, r_3, \ldots}^{(r)} \prod_{k=1}^{\infty} \frac{1}{r_k!} \left(\eta_k\right)^{r_k}. \tag{22}$$

Here r_k is the number of k-contractions, and the sum $\sum^{(r)}$ is subject to the constraint $\sum_k k r_k = r$. To obtain Eq. (22), we observe that

$$\left(\sum_{\langle i_1, i_2, \ldots, i_k \rangle}^{(\text{in})} \| H^{(k)}_{i_1, i_2, \ldots i_k} \| \right)^{r_k} \tag{23}$$

contains each way of choosing r_k k-contractions among the r marked locations $r_k!$ times, plus additional nonnegative terms; the factor $1/r_k!$ in Eq. (22) compensates for this overcounting.

To go further we wish to relate η_k for $k > 1$ to η_1. Note that in $\eta_k^{(k+l)}$ for $k > 1$, we can replace the sum over ways to choose k qubits by a sum over all qubits divided by $k!$, and similarly we can replace the sum over the ways to choose l qubits by a sum over all qubits divided by $l!$, obtaining

$$\eta_k^{(k+l)} = \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k}^{(\text{in})} \frac{1}{l!} \sum_{j_1, j_2, \ldots, j_l}^{(\text{all})} \| H^{(k+l)}_{i_1, i_2, \ldots i_k, j_1, j_2, \ldots, j_l} \| \; t_0. \tag{24}$$

By summing i_i over the r marked locations we obtain the bound

$$\eta_k^{(k+l)} \le r \ \max_{i_1}^{(\mathrm{in})} \frac{1}{k!} \sum_{i_2,\ldots,i_k}^{(\mathrm{all})} \frac{1}{l!} \sum_{j_1,j_2,\ldots,j_l}^{(\mathrm{all})} \|H^{(k+l)}_{i_1,i_2,\ldots i_k,j_1,j_2,\ldots,j_l}\| \ t_0; \tag{25}$$

note that we still have a bound if we extend the "in" sum to a sum over all qubits. From Eq. (19) we have

$$\eta_1^{(k+l)} = \max_{i_1}^{(\mathrm{in})} \frac{1}{(k+l-1)!} \sum_{i_2,\ldots,i_k}^{(\mathrm{all})} \sum_{j_1,j_2,\ldots,j_l}^{(\mathrm{all})} \|H^{(k+l)}_{i_1,i_2,\ldots i_k,j_1,j_2,\ldots,j_l}\| \ t_0. \tag{26}$$

which implies (for $k > 1$)

$$\eta_k^{(k+l)} \le r\frac{(k+l-1)!}{k!\ l!}\eta_1^{(k+l)} = \frac{r}{k}\binom{k+l-1}{l}\eta_1^{(k+l)} \le \frac{r}{k}2^{k+l-1}\eta_1^{(k+l)}. \tag{27}$$

Hence we find (for $k > 1$)

$$\eta_k \le \frac{r}{2k}\sum_{l=0}^{\infty} 2^{k+l}\eta_1^{(k+l)}. \tag{28}$$

Now suppose, as in the hypothesis of Theorem 1, that

$$\eta_1^{(k)} \le \frac{1}{(k-1)!}\ \tilde{\eta}_1^{(k)} \le \frac{f_k\alpha^k}{(k-1)!}. \tag{29}$$

From Eq. (28) we obtain

$$\eta_k \le \frac{rg_k}{2k!}(2\alpha)^k, \tag{30}$$

where

$$g_k = f_k + \sum_{l=1}^{\infty} \frac{(k-1)! f_{k+l}(2\alpha)^l}{(k+l-1)!}. \tag{31}$$

Then the bound Eq. (22) becomes

$$\varepsilon^r \le \sum_{r_1,r_2,r_3,\ldots}^{(r)} \prod_{k=1}^{\infty} \frac{1}{r_k!}\left(\frac{rg_k(2\alpha)^k}{2k!}\right)^{r_k} = (2\alpha)^r \sum_{r_1,r_2,r_3,\ldots}^{(r)} \prod_{k=1}^{\infty} \frac{1}{r_k!}\left(\frac{rg_k}{2k!}\right)^{r_k}, \tag{32}$$

recalling the constraint on the sum. If we now relax the constraint on the sum, we have

$$\begin{aligned} \varepsilon^r \quad &\le \quad (2\alpha)^r \prod_{k=1}^{\infty}\sum_{r_k=0}^{\infty} \frac{1}{r_k!}\left(\frac{rg_k}{2k!}\right)^{r_k} = (2\alpha)^r \prod_{k=1}^{\infty} \exp\left(\frac{rg_k}{2k!}\right) \\ &= \quad (2\alpha)^r \left(\exp\left(\sum_{k=1}^{\infty}\frac{g_k}{2k!}\right)\right)^r = \left(2\alpha\ \exp\left(\sum_{k=1}^{\infty}\frac{g_k}{2k!}\right)\right)^r. \end{aligned} \tag{33}$$

Finally, we observe that in the case of an m-qubit gate location, the location is faulty if the system-bath perturbation acts nontrivially on any one of m qubits, which enhances the noise strength by the factor m. This completes the proof of Theorem 1. □

If the hypotheses of Theorem 1 fail, we need not despair. In [NP09] it is shown that scalability may still be provable if we make further assumptions about the *state* of the bath (which was assumed there to be a Gaussian state in which spatial correlations decay sufficiently rapidly). We might instead suppress the noise correlations using specialized methods that are not incorporated into the standard proofs of the quantum threshold theorem, such as dynamical decoupling. Characterizing the residual noise correlations when dynamical decoupling is employed seems to be a hard problem, though some preliminary steps were taken in [NLP11].

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