Ph 106a Final Exam
Due: Thursday 13 December 2003, 4pm

- This exam is to be taken in one continuous time interval not to exceed **4 hours**, beginning when you first open the exam. (You may take one 15 minute break during the exam, which does not count as part of the 4 hours.)

- You may consult the textbook *Classical Mechanics* by Goldstein, Professor Golwala's on-line lecture notes (there is a link to the notes on the Ph 106a website), your lecture notes, and the problem sets and solutions. If you wish, you may use a calculator, computer, or integral table for doing calculations. However, this probably won't be necessary. **No other materials or persons are to be consulted.**

- There are four problems, each with multiple parts, and 200 possible points; the value of each problem is indicated. You are to work all of the problems.

- The completed exam is to be deposited in the box outside 448 Lauritsen, no later than 4:00 pm on Thursday 13 December 2007. **No late exams will be accepted.**

- Good luck!

1. Rolling cyllinders — 50 total points

A solid cylinder of radius r and mass m rolls *without slipping* inside a larger hollow cylinder of radius R. The axes of the cylinders are horizontal and parallel. The motion takes place in the uniform gravitational field of the earth. The larger cylinder is driven so that it rotates about its axis with constant angular *acceleration* α. (See Figure 1.)

a) (20 points) Defining θ as shown in Figure 1, express the kinetic energy of the smaller cylinder in terms of $\dot{\theta}$. (You must use the constraint of rolling without slipping to express the angular velocity of the solid cylinder in terms of $\dot{\theta}$ and the angular velocity $\omega = \alpha t$ of the hollow cylinder.)

b) (20 points) Find the equation of motion for θ? (Write down the Lagrangian and derive the Euler-Lagrange equation.)

c) (10 points) Find a condition which must be satisfied by α so that an equilibrium value of θ exists, and find the equilibrium value of θ, assuming the condition is satisfied.

2. Hanging masses — 50 total points

A block of mass $2m$ hangs vertically from a fixed ceiling by a spring with spring constant k and unstretched length L. A second block of mass m hangs from the first block by an identical spring. The blocks hang in a uniform gravitational field with gravitational acceleration g. (See Figure 2.)

a) (15 points) Express the Lagrangian of the system in the form

$$L = \frac{1}{2} \sum_{i,j} \left(T_{ij} \ \dot{\eta}_i \dot{\eta}_j - V_{ij} \ \eta_i \eta_j \right)$$

where η_1 and η_2, are the the vertical displacements of the two blocks from their equilibrium positions. (Assume that only vertical motion takes place.)

b) (20 points) Find the frequencies ω_1, ω_2 of the normal modes of oscillation about the equilibrium position.

c) (15 points) Find the (normalized) normal mode vectors $\vec{X}^{(1)}$ and $\vec{X}^{(2)}$ such that

$$\eta_i = \sum_a X_i^{(a)} Q_a$$

and

$$L = \frac{1}{2} \sum_a \left(\dot{Q}_a^2 - \omega_a^2 Q_a^2 \right)$$

(Recall that the normal mode vectors satisfy the normalization condition $\sum_{ij} X_i^{(a)} T_{ij} X_j^{(b)} = \delta^{ab}$.)

3. Canonical transformation — 50 total points

a) (10 points) Use the Poisson bracket to show that the transformation

$$Q = q\cos\alpha - p\sin\alpha$$

$$P = q\sin\alpha + p\cos\alpha$$

is canonical, for any value of α.

b) (25 points) Find a generating function for this transformation of the form $G_2(q, P)$.

c) (15 points) If the Hamiltonian is invariant under this transformation, what function of q and p is conserved? (Consider the form of G_2 for α infinitesimal.)

4. Another Hamilton-Jacobi theory — 50 total points

In the version of Hamilton–Jacobi theory developed in class and in the textbook, a generating function of the type $G_2(q, P, t)$ is sought which generates a canonical transformation to new canonical variables Q and P which are constants of the motion. But the choice of this type of generating function was arbitrary. In this problem, you are to pursue the alternative of finding a generating function of the type $G_3(Q, p, t)$.

a) (15 points) Consider a system with one degree of freedom and Hamiltonian $H(q, p)$. Write down a Hamilton–Jacobi partial differential equation for a function $T(p, t)$, such that the solution $T(Q, p, t)$ (depending on the one constant of integration Q) generates a canonical transformation to constant variables Q and P.

b) (5 points) For a particle moving vertically in a uniform gravitational field, the Hamiltonian is

$$H = \frac{1}{2m}p^2 + mgq.$$

What is the Hamilton–Jacobi equation for $T(p, t)$ in this case?

c) (15 points) Find a solution to this equation of the form

$$T(p, t) = W(p) - Qt$$

d) (15 points) Find the canonical transformation generated by $T(Q, p, t)$. That is, express Q and P in terms of q, p, and t, and invert these expressions to find q and p in terms of Q, P and t.

Figure 1

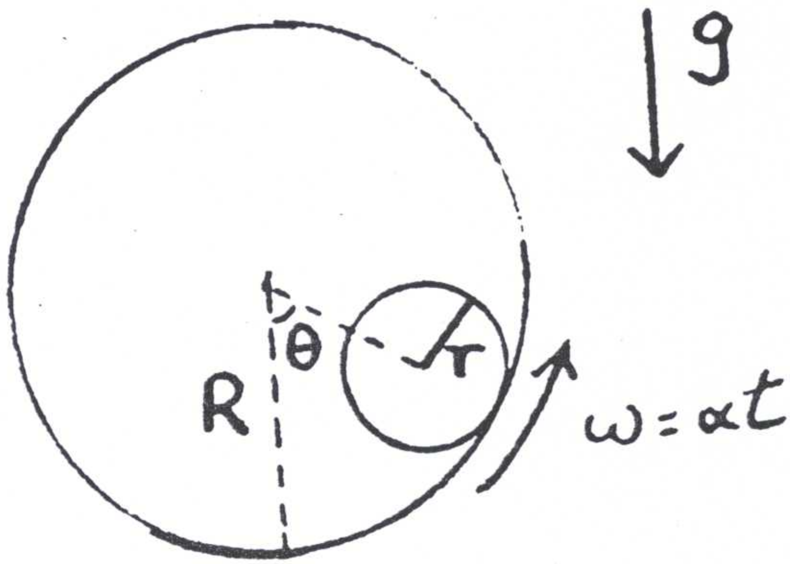

Figure 2