Ph 106a Midterm Exam
Due: Thursday 8 November 2007, 4pm

- This exam is to be taken in one continuous time interval not to exceed **3 hours**, beginning when you first open the exam.

- You may consult the textbook *Classical Mechanics* by Goldstein, your lecture notes, and the problem sets and solutions. If you wish, you may use a calculator, computer, or integral table for doing calculations. However, this probably won't be necessary. **No other materials or persons are to be consulted.**

- There are two problems, each with multiple parts, and 100 possible points; the value of each problem is indicated. You are to work all of the problems.

- The completed exam is to be deposited in the box outside 448 Lauritsen, no later than 4pm on Thursday 8 November 2007. **No late exams will be accepted.**

- Good luck!

1. Motion in perpendicular electric and magnetic fields — 50 points

A particle of charge q and mass m starts from rest at time $t = 0$ at the origin $(0, 0, 0)$ of a Cartesian coordinate system. There are constant electric and magnetic fields throughout all of space, given by

$$\vec{E} = E_0 \ \hat{e}_y \ ,$$

$$\vec{B} = B_0 \ \hat{e}_z \ .$$

See Fig. 1.

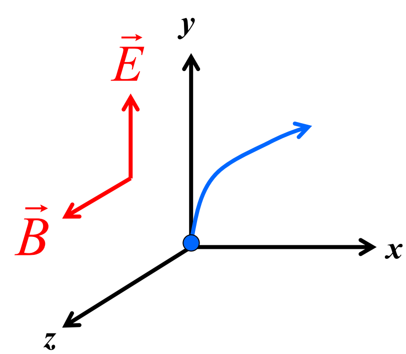

Figure 1: A charged particle moving in an electric and magnetic field.

(a) (5 points) Show that these fields can be described by the potentials

$$\phi = -E_0 y \ , \qquad \vec{A} = -B_0 y \ \hat{e}_x \ .$$

(b) (5 points) What is the Lagrangian for the system?

(c) (15 points) Identify three symmetries, and find the three corresponding constants of the motion. Evaluate these constants by using the initial conditions.

(d) (10 points) Use the conservation laws to express each component of the velocity $\big(\dot{x}(t), \dot{y}(t), \dot{z}(t)\big)$ in terms of y.

(e) (15 points) Find the trajectory $\big(x(t), y(t), z(t)\big)$.

Hint: An integral of the form

$$\int dx [x(a-x)]^{-1/2}$$

can be performed by substituting

$$x = \frac{1}{2}a(1 - \cos\theta) \ .$$

2. The wobbling coin — 50 points

When a coin is dropped on a horizontal table, it usually settles down only after a wobbling motion. In this motion, the center of the coin remains approximately at rest (except for a slow settling due to loss of energy), and the edge of the coin rolls on the surface, while the axis normal to the coin precesses rapidly about the vertical.

Suppose that a coin of mass m, and radius R, with uniform mass density and negligible thickness, executes this motion. Assume that the edge of the coin rolls without slipping on the table, that energy is conserved, and that the angle between the normal to the coin and the vertical is θ. The problem is to calculate the circular frequency ω_p of the precession in terms of R, θ, and g (where g is the acceleration due to gravity).

(a) (10 points) Let $\{\hat{e}'_1, \hat{e}'_2, \hat{e}'_3\}$ denote the three principal axes of the coin through its center of mass, where $\hat{e}'_3$ points perpendicular to the plane of the coin, and $\hat{e}'_1, \hat{e}'_2$ lie in the plane of the coin. Find the moments of inertia I_1, I_2, I_3 of the coin for rotation about these axes.

(b) (10 points) The angular velocity of the coin can be expressed as the sum of two components:

$$\vec{\omega} = \omega_r \hat{e}'_3 + \omega_p \hat{e}_3 \ ,$$

where $\hat{e}_3$ is the upward-pointing unit vector normal to the table, and $\hat{e}'_3$ is the body-fixed axis normal to the coin. Thus, the coin rotates about its axis of symmetry with circular frequency ω_r, and precesses about the vertical with circular frequency ω_p. From the condition that the coin rolls without slipping, derive a relation between ω_r and ω_p. Recall that if the coin rolls without slipping, then the point on the edge of the coin that is in contact with the table is instantaneously at rest. Assume that the center of mass of the coin is also at rest.

(c) (10 points) It is convenient to analyze the motion of the coin in a rotating reference frame (*not* the same as the body-fixed frame) that rotates about the vertical with angular velocity

$$\vec{\omega}_{\text{frame}} = \omega_p \hat{e}_3 \ ;$$

In this rotating frame, the coin does not precess, so that both the angular velocity and angular momentum of the coin are time-independent vectors. Choose the basis vectors for the rotating frame to be $\{\hat{e}_1^{(a)}, \hat{e}_2^{(a)}, \hat{e}_3^{(a)}\}$, where (as shown in Fig. 2), $\hat{e}_3^{(a)} = \hat{e}'_3$ is aligned with the coin's symmetry axis, and $\hat{e}_1^{(a)}$ points from the center of mass toward the point P where the coin is in contact with the table. Find the components of the coin's angular velocity $\vec{\omega}$ and its angular momentum $\vec{L}$ in this rotating coordinate system.

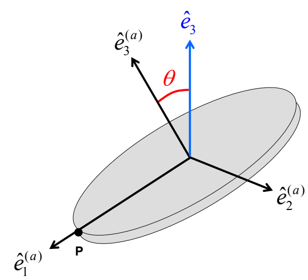

Figure 2: The wobbling coin, in the rotating coordinate frame $\{\hat{e}_1^{(a)}, \hat{e}_2^{(a)}, \hat{e}_3^{(a)}\}$ that freezes the precession of the coin's symmetry axis. The unit vector $\hat{e}_3^{(a)}$ points along the symmetry axis, and the unit vector $\hat{e}_1^{(a)}$ points toward the point P where the coin is in contact with the table.

(d) (5 points) In the rotating reference frame described in (c) above, find the components of the torque $\vec{N}$ acting on the coin about its center of mass. (Note that because the center of mass of the coin is at rest, the upward force exerted by the table on the coin at the point of contact P must be equal and opposite to the downward gravitational force exerted through the coin's center of mass.)

(e) (15 points) In the rotating reference frame described in (c), write out the components of the equation of motion

$$\vec{N} = \left(d\vec{L}/dt\right)_{\text{space}} = \left(d\vec{L}/dt\right)_{\text{frame}} + \vec{\omega}_{\text{frame}} \times \vec{L} = \vec{\omega}_{\text{frame}} \times \vec{L} \ .$$

(Note that $\left(d\vec{L}/dt\right)_{\text{frame}} = 0$, because $\vec{L}$ is constant in the rotating frame.) From this equation, find the circular precession frequency ω_p, in terms of R, θ, and g.