Ph 106a

Homework Assignment No. 3
Due: Thursday, 25 October 2007

1. Galilean relativity. Another important symmetry of classical mechanics, not mentioned in lecture, is called a "boost." It is the transformation to a new coordinate system moving with uniform velocity relative to the original coordinate system:

$$\vec{x}'_a = \vec{x}_a + \varepsilon \hat{n} t \ . \tag{1}$$

Show that this is a symmetry for a system of (nonrelativistic) particles with interactions described by potentials that depend only on the distances between the particles, and find the conservation law associated with this symmetry. Give a physical interpretation of this conservation law. (The three independent boosts, three rotations, three spatial translations, and time translation form the ten-parameter symmetry group of classical mechanics, which is known as the Galilean group.)

2. Precession of the orbit of Mercury.

a) Suppose that an inverse square law of force is modified by a small perturbation, so that the potential energy is given by

$$V = -\frac{k}{r}\Big(1 + v(r)\Big) \ , \tag{2}$$

where $v(r) \ll 1$ and $rv'(r) \ll 1$. Find the frequency ω_r of the small oscillations of the radius of a particle of mass m in a nearly circular orbit with angular momentum ℓ and radius r_0. Thus find the frequency ω_p of precession of the perihelion of a nearly circular orbit,

$$\omega_p = \omega_\theta - \omega_r \ , \tag{3}$$

where ω_θ is the angular frequency.

b) The Newtonian law of universal gravitation is modified by general relativity to become, for a particle of mass m moving slowly ($|\vec{v}| \ll c$) under the influence of a much heavier particle ($M \gg m$),

$$V = -\frac{GMm}{r} - \frac{R\ell^2}{2mr^3} \tag{4}$$

for $r \gg R$. The quantity $R = 2GM/c^2$ is called the gravitational radius of the particle with mass M. Assuming that the orbit of Mercury is nearly circular with radius r_0, estimate the angle by which the perihelion of Mercury's orbit should precess in 100 years. Take

$$\begin{aligned} GM &= 1.33 \times 10^{20} (\text{meter})^3 (\text{sec})^{-2} \\ r_0 &= 5.79 \times 10^{10} \text{ meter} . \end{aligned} \tag{5}$$

The eccentricity of Mercury's orbit is actually $e = 0.206$, which causes a correction by the factor $(1 - e^2)^{-1}$ to your answer. The observed value is 43.1 ± 0.4 seconds of arc.

3. To Paris in a hurry. Suppose that an international subway system is designed in which cars slide through frictionless tubes using only the gravitational force. Find a *first-order* differential equation, involving one constant of integration, for the trajectory the tube should follow so that a car reaches its destination in the minimum possible time. You may ignore the rotation of the earth. As measured from the center of the earth, the initial and final points are separated by an angle θ_0. Assume that the trajectory lies in the plane containing these three points. (You don't need to solve the differential equation; just write it down.)