Ph 106a

Homework Assignment No. 5
Due: Thursday, 15 November 2007

1. Sleeping top. In class we considered the "sleeping" top, the symmetrical top with one point fixed and initial conditions $\theta = \dot{\theta} = 0$. We saw that if the angular velocity ω_s of the top exceeds the critical value $\omega_{\rm crit} = (2/I_3)(Mg\ell I)^{1/2}$, then the vertical position of the top is stable.

a) If a sleeping top with $\omega_s > \omega_{\rm crit}$ receives a small horizontal impulse, then the symmetry axis of the top will execute small oscillations about the vertical. That is, the inclination θ from the vertical will vary as

$$\theta \;\approx\; \theta_0 \cos(\omega t) \; . \tag{1}$$

where θ_0 is a small constant that depends on the size of the impulse. Express the circular frequency ω of these oscillations in terms of ω_s, $\omega_{\rm crit}$, I_3, and I.

b) When the (stable) vertical sleeping top receives a small horizontal impulse, it returns to the vertical after time $t_0 = \pi/\omega$. By what amount $\Delta\phi$ does the Euler angle ϕ (the azimuthal position of the top's symmetry axis) advance in the time t_0? What is $\Delta\phi$ in the limit of a fast top with $\omega_s \gg \omega_{\rm crit}$?

2. Sliding top. Suppose the axis of a symmetrical top ends at a contact point that slides without friction on a horizontal table. The top has mass M, ℓ is the distance from the contact point to the top's center of mass, I_3 is the top's moment of inertia for rotations about it's symmetry axis, and I is the moment of inertia for rotations about an axis perpendicular to the symmetry axis that passes through the top's center of mass.

a) Using as generalized coordinates the x and y coordinates of the top's center of mass, and the three Euler angles θ, ϕ, ψ describing rotations of the top about it's center of mass, write down the Lagrangian for this system.

b). Using conservation laws, find a function $g(u)$ such that

$$\dot{u}^2 = g(u), \tag{2}$$

where $u = \cos\theta$.

3. Small oscillations. For the mechanical systems (i) and (ii) illustrated on page 3:

a) Choose suitable generalized coordinates and find the Lagrangian.

b) Find the equilibrium positions.

c) Expand to quadratic order to find matrices T_{ij} and V_{ij} such that the Lagrangian can be expressed as

$$L = \frac{1}{2}\sum_{i,j} \left(T_{ij}\ \dot{\eta}_i\dot{\eta}_j - V_{ij}\ \eta_i\eta_j\right) + \cdots \ , \tag{3}$$

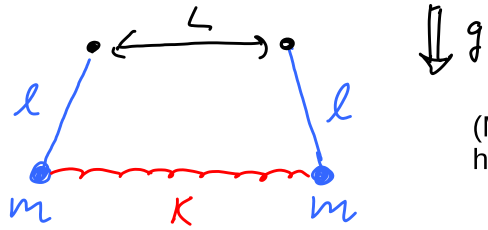
where η_i is the deviation of the ith generalized coordinate from its equilibrium position.

(For each spring, the natural unstretched length is L_0.)

4. More small oscillations. Same as problem 3, for mechanical system (iii) illustrated on page 3.

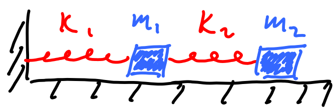
Illustrations for Problems 3 and 4:

(i)



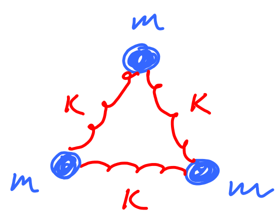
(Motion confined to the plane. Pivots of both pendula are at the same height.)

(ii)



(Motion confined to one dimension.)

(iii)



(Motion confined to the plane.)