Ph 106a

Homework Assignment No. 6
Due: Thursday, 29 November 2007

1. **Normal modes**. For the mechanical system (ii) of problem 3 on Homework No. 5, find the normal mode frequencies ω_a, the normal mode vectors $\vec{X}^{(a)}$, and the normal coordinates Q_a, such that

$$\eta_i = \sum_a X_i^{(a)} Q_a \tag{1}$$

and

$$\begin{aligned} L &= \frac{1}{2}\sum_{i,j} (T_{ij}\ \dot{\eta}_i\dot{\eta}_j - V_{ij}\ \eta_i\eta_j) \\ &= \frac{1}{2}\sum_a (\dot{Q}_a^2 - \omega_a^2 Q_a^2)\ . \end{aligned} \tag{2}$$

Verify that the normal mode vectors obey the orthogonality condition

$$\sum_{i,j} X_i^{(a)} T_{ij} X_j^{(b)} = \delta^{ab}\ . \tag{3}$$

Define your generalized coordinates η_i as in the solution to last week's problem 3, and use the matrices T_{ij} and V_{ij} found in that solution.

2. **Masses and springs**. Same as problem (1) above, but for the mechanical system (iii) of problem 4 on Homework No. 5.

3. **Action as a function of the coordinates**. For a free particle of mass m in one dimension, calculate the action S of a trajectory that solves the equation of motion, where the particle begins at position x_1 at time t_1 and ends at position x_2 at time t_2. Verify the relations derived in class:

$$\frac{\partial S}{\partial x_2} = p(t_2)\ ,\ \frac{\partial S}{\partial t_2} = -H(t_2)\ . \tag{4}$$

4. Virial theorem. Consider a gas of N particles which interact gravitationally in three dimensions. Assume that the gas is in equilibrium so that

$$\left\langle \frac{d}{dt} \sum_i p_i q_i \right\rangle = 0, \tag{5}$$

where $\langle \cdot \rangle$ denotes the time average. Use the virial theorem in the form

$$\langle T \rangle = \frac{1}{2} \left\langle \sum_i q_i \frac{\partial H}{\partial q_i} \right\rangle \tag{6}$$

to find a relation between the time-averaged quantities $\langle T \rangle$ and $\langle V \rangle$, where the potential energy V is defined so that it vanishes when each particle is infinitely far from all the others.