Ph 106a

Homework Assignment No. 7
Due: Thursday, 6 December 2007

1. Adiabatic invariance. Consider a much simplified model of a gas which consists of a single free particle of mass m moving in one dimension between walls located at $x = 0$ and $x = L$. The walls can be treated as infinitely high potential barriers.

a) Calculate the action variable J as a function of L and the energy E of the particle. Verify that the period of the motion is given by $T = dJ/dE$.

b) If the particle moves very fast, it makes sense to speak of the "average force" $\langle F \rangle$ exerted on the walls by the particle. Express $\langle F \rangle$ as a function of L and E, and use the adiabatic invariance of the action variable to find a relation between $\langle F \rangle$ and L which holds when L is changed slowly.

2. Liouville's theorem. Consider a "lens" which has been designed to "focus" a two-dimensional beam of particles. Particles entering the lens from the left all have the same value of p_x, but have a spread of y-momentum ranging between $-\frac{1}{2}\Delta p_y$ and $\frac{1}{2}\Delta p_y$. The entrance opening of the lens has height L_1. Inside the lens, forces are exerted in the y direction to compress the beam, so that it exits through an opening of height $L_2 < L_1$. If the interactions between different particles in the beam may be ignored, the beam can be regarded as an ensemble of systems with the same Hamiltonian, and Liouville's theorem applies. Use Liouville's theorem to derive an inequality which must be satisfied by $(\Delta p_y)_{\text{final}}$, the final spread of y-momenta.

3. Canonical transformation. Find the values of α and β for which the transformation

$$\begin{aligned} Q &= q^{\alpha}\ \cos\beta p\ , \\ P &= q^{\alpha}\ \sin\beta p\ , \end{aligned} \tag{1}$$

is canonical. For these values of α and β, construct a generating function $G_3(p, Q)$ for the canonical transformation.

4. Hamilton-Jacobi equation. Show that the Hamilton-Jacobi equation is separable for a particle moving in a potential of the form

$$V(r,\theta,\phi) = V_r(r) + \frac{V_\theta(\theta)}{r^2} + \frac{V_\phi(\phi)}{r^2 \sin^2\theta} \ , \tag{2}$$

where r, θ, ϕ are spherical coordinates. Find the general solution for $S(q, P, t)$, and from it derive the general solution to the equations of motion.