Ph 12b Midterm Exam
Due: Wednesday, 10 February 2010, 5pm

- This exam is to be taken in one continuous time interval not to exceed **3 hours**, beginning when you first open the exam.

- You may consult the textbook *Introductory Quantum Mechanics* by Liboff, the textbook *Introduction to Quantum Mechanics* by Griffiths, your lecture notes, the online lecture notes, and the problem sets and solutions. If you wish, you may use a calculator, computer, or integral table for doing calculations. However, this probably won't be necessary. **No other materials or persons are to be consulted.**

- There are three problems, each with multiple parts, and 100 possible points; the value of each problem is indicated. You are to work all of the problems.

- The completed exam is to be handed in at the Ph 12 in-box outside 264 Lauritsen. All exams are due at 5pm on Wednesday, February 10. **No late exams will be accepted.**

- Good luck!

1. Two-state quantum dynamics — 35 total points

Let $|e_1\rangle$ and $|e_2\rangle$ denote two normalized and mutually orthogonal states in a Hilbert space $\mathcal{H}$: $\langle e_1|e_1\rangle = \langle e_2|e_2\rangle = 1,\ \langle e_1|e_2\rangle = 0$. A certain quantum system has Hamiltonian $\hat{H}$, and the two normalized states

$$\begin{aligned} |\omega_1\rangle &= \frac{1}{2}|e_1\rangle + \frac{\sqrt{3}}{2}|e_2\rangle \ , \\ |\omega_2\rangle &= \frac{-\sqrt{3}}{2}|e_1\rangle + \frac{1}{2}|e_2\rangle \ , \end{aligned}$$

are eigenstates of $\hat{H}$ with eigenvalues $\hbar\omega_1, \hbar\omega_2$ respectively:

$$\begin{aligned} \hat{H}|\omega_1\rangle &= \hbar\omega_1|\omega_1\rangle \ , \\ \hat{H}|\omega_2\rangle &= \hbar\omega_2|\omega_2\rangle \ . \end{aligned}$$

At time $t = 0$, the system is prepared in the state $|\psi(0)\rangle = |e_1\rangle$.

(*a*) (10 points) Express $|\psi(0)\rangle$ as a linear combination of energy eigenstates.

(*b*) (5 points) Solve the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

to find the state $|\psi(t)\rangle$ at time t. Express your answer in the form

$$|\psi(t)\rangle = f_1(t)|\omega_1\rangle + f_2(t)|\omega_2\rangle \ .$$

(*c*) (10 points) Now re-express $|\psi(t)\rangle$ in the form

$$|\psi(t)\rangle = g_1(t)|e_1\rangle + g_2(t)|e_2\rangle \ .$$

(*d*) (10 points) The states $|e_1\rangle$ and $|e_2\rangle$ are eigenstates of an observable $\hat{A}$:

$$\begin{aligned} \hat{A}|e_1\rangle &= a_1|e_1\rangle \ , \\ \hat{A}|e_2\rangle &= a_2|e_2\rangle \ , \end{aligned}$$

where $a_1 \neq a_2$, that is measured at time t. Find the probability $P(a_1)$ that the outcome of the measurement is a_1 and the probability $P(a_2)$ that the outcome is a_2.

2. Particle in a box — 35 total points

A free quantum-mechanical particle with mass m moves inside a one-dimensional box with impenetrable walls located at $x = \pm a/2$.

(*a*) (10 points) Find the normalized wave function $\psi_0(x)$ of the energy eigenstate of lowest energy, and the normalized wave function $\psi_1(x)$ of the energy eigenstate of next-to-lowest energy. Find also the corresponding energy eigenvalues E_0 and E_1.

(*b*) (5 points) Suppose that at time $t = 0$ the particle is in the state with wave function $\psi(x,0) = \sqrt{\frac{1}{3}}\,\psi_0(x) + \sqrt{\frac{2}{3}}\,\psi_1(x)$. What is the wave function $\psi(x,t)$ at the subsequent time t?

(*c*) (20 points) Find the expectation value

$$\langle \psi(t)|\hat{x}|\psi(t)\rangle$$

of the position operator $\hat{x}$ at time t. (**Hint**: An integral of the form

$$\int dx\ x\ \sin(Ax)$$

can be done using integration by parts.)

3. Two qubits — 30 total points

A qubit is a quantum system whose Hilbert space is two dimensional. Consider two qubits labeled A and B, where $\{|e_0\rangle, |e_1\rangle\}$ is a basis for qubit A, and $\{|f_0\rangle, |f_1\rangle\}$ is a basis for qubit B. Suppose the state vector for the composite system AB is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(\cos\theta|e_0\rangle \otimes |f_0\rangle + \sin\theta|e_0\rangle \otimes |f_1\rangle + \sin\theta|e_1\rangle \otimes |f_0\rangle + \cos\theta|e_1\rangle \otimes |f_1\rangle\big).$$

(*a*) (15 points) Alice has no access to qubit B; she can perform measurements only on qubit A. For any measurement that Alice might perform on qubit A, the probability distribution for the measurement outcomes is determined by the density operator $\hat{\rho}$ for qubit A. Express this density operator as a 2×2 matrix in the basis $\{|e_0\rangle, |e_1\rangle\}$.

(*b*) (5 points) Suppose Alice performs an orthogonal measurement in the basis $\{|e_0\rangle, |e_1\rangle\}$. What is the probability that Alice's outcome is $|e_0\rangle$?

(*c*) (10 points) Now suppose that Bob, who *does* have access to qubit B, measures his qubit in the basis $\{|f_0\rangle, |f_1\rangle\}$, and reports to Alice that he obtained the outcome $|f_0\rangle$. After learning of Bob's outcome, Alice performs her measurement in the basis $\{|e_0\rangle, |e_1\rangle\}$. In this case, what is the probability that Alice's outcome is $|e_0\rangle$?