# Ph 12b

**Homework Assignment No. 1**
**Due: 5pm, Thursday, 14 January 2010**

**1. Uncertainty principle and the quantum harmonic oscillator**. For any quantum state of a single particle moving in one dimension, there is a corresponding probability density $P(x)$ that governs the possible outcomes when the position of the particle is measured. $P(x)$ is a nonnegative function normalized so that

$$\int_{-\infty}^{\infty} dx\ P(x) = 1,$$

and the expectation value $\langle f \rangle$ of the function $f(x)$ is

$$\langle f \rangle = \int_{-\infty}^{\infty} dx\ P(x) f(x).$$

The standard deviation $\Delta x$ of the position from its mean, defined as

$$(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle,$$

is called the position *uncertainty*. Similarly, another probability density $Q(p)$ associated with the same quantum state governs the outcomes when the momentum of the particle is measured; we may use $Q(p)$ to compute expectation values of functions of $p$, and $\Delta p$, the standard deviation of the momentum from its mean, is the momentum uncertainty. The position and momentum uncertainties are related by Heisenberg's uncertainty principle,

$$\Delta x \Delta p \geq \hbar/2.$$

*a*) The energy $E$ of a harmonic oscillator with mass $m$ and circular frequency $\omega$ can be expressed as

$$E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2.$$

For a quantum state of the oscillator with position uncertainty $\Delta x$ and $\langle x \rangle = 0 = \langle p \rangle$, use the uncertainty principle to find a lower bound on $\langle E \rangle$, expressed in terms of $\Delta x$.

*b*) Now find the value of $\Delta x$ that minimizes your lower bound from part (*a*), and derive a lower bound on $\langle E \rangle$ that applies to any quantum state. As you will learn later in this course, the ground-state energy of a one-dimensional harmonic oscillator is $E_0 = \hbar\omega/2$. Compare this value to your lower bound.

**2. Uncertainty principle and the standard quantum limit for position measurement**. Suppose that the position of a free particle in one dimension is measured at time zero, then measured once again at time $t$. A particle with momentum $p$ has velocity $v = p/m$; therefore the particle position $x_t$ right before the second measurement is related to the particle position $x_0$ and momentum $p_0$ right after the first measurement by

$$x_t = x_0 + p_0 t/m.$$

*a*) Suppose that the quantum state right after the first position has position uncertainty $\Delta x_0$, and consider $\Delta x_t$, the standard deviation from its mean of $x_t$ evaluated using this quantum state. Invoke the uncertainty principle to obtain a lower bound on $\Delta x_t$ expressed in terms of $\Delta x_0$. In your derivation, assume that the position and momentum of the particle are "uncorrelated" right after the first measurement, so that

$$\langle\, (x_0 - \langle x_0 \rangle)\,(p_0 - \langle p_0 \rangle)\,\rangle + \langle\, (p_0 - \langle p_0 \rangle)\,(x_0 - \langle x_0 \rangle)\,\rangle = 0.$$

*b*) What value of $\Delta x_0$ minimizes your lower bound on

$$\Delta x_0 \Delta x_t?$$

Find a lower bound on $\Delta x_0 \Delta x_t$ that applies for any value of $\Delta x_0$. The square root of your result is called the *standard quantum limit* on repeated position measurement for a free mass.

*c*) At the Laser Interferometer Gravitational-Wave Observatory (LIGO), repeated measurements separated by time interval $t = 10^{-2}$ have been performed for a 10 kg free mass, with sensitivity less than a factor of 10 above the standard quantum limit. In meters, what is this standard quantum limit? Compare to the size of a proton.

(Wow!) In about five years, Advanced LIGO is expected to achieve a sensitivity 10 times better for a mass 4 times heavier.

**3. Old quantum theory and the harmonic oscillator**.

*a*) The Hamiltonian of a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

What are Hamilton's equations of motion for this system?

*b*) The energy $E$ of the oscillator is a constant of the motion, and the condition $H = E$ determines a closed elliptical orbit in phase space (*i.e.*, the $x{-}p$ plane). Sketch the orbits for several different value of the energy $E$, indicating the direction of flow along the orbit determined by Hamilton's equations.

*c*) For a closed orbit in the two-dimensional $x{-}p$ phase space, the *action* $J$ is defined as the area enclosed by the orbit. Compute $J(E)$ for the harmonic oscillator with energy $E$. (Recall that the area $A$ of an ellipse with semi-major axis $a$ and semi-minor axis $b$ is $A = \pi ab$.)

*d*) A theorem of classical mechanics asserts that for periodic motion in one dimension, the action is related to the period $T$ of the motion by

$$T = \partial J / \partial E.$$

Verify this relation for the harmonic oscillator.

*e*) According to the "old quantum theory" a one-dimensional periodic system has a discrete set of allowed energy levels $\{E_n\}$, determined by the requirement that the action $J$ is an integer multiple of Planck's constant $h = 2\pi\hbar$:

$$J(E_n) = nh, \quad n = 0, 1, 2, 3, \ldots.$$

(For this reason, some older books refer to $h$ as the "quantum of action.") Use this rule to find the energy levels of the harmonic oscillator.

**4. Poisson brackets**. Consider a Hamiltonian system with coordinates $\{q_a, a = 1, 2, \ldots, N\}$ and conjugate momenta $\{p_a, a = 1, 2, \ldots, N\}$, and let $A(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N)$ and $B(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N)$ be functions of these variables. The *Poisson bracket* $[A, B]$ of $A$ and $B$ is defined as

$$[A, B] = \sum_{a=1}^{N} \left( \frac{\partial A}{\partial q_a} \frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial q_a} \frac{\partial A}{\partial p_a} \right)$$

*a*) Use Hamilton's equations to show that

$$\frac{d}{dt}A = [A, H],$$

where $H$ is the Hamiltonian.

*b*) If $B = B(A)$ is a function of $A$, show that $[A, B] = 0$

*c*) Use the Poisson bracket to show that, if the Hamiltonian $H$ has no explicit dependence on time (and hence is a function of only the phase space variables), then $dH/dt = 0$ (*i.e.*, energy is conserved).

*b*) Evaluate the Poisson brackets

$$[q_a, q_b], \quad [p_a, p_b], \quad [q_a, p_b],$$

for $a, b = 1, 2, 3, \ldots N$.

**5. Single-photon interference**. Consider an idealized version of a double-slit interference experiment, in which a single photon can pass through either one of two slits in a screen, labeled $A$ and $B$, and can be detected behind the screen by either one of two detectors, labeled $C$ and $D$. If the photon passes through slit $A$, then the amplitude for the photon to arrive at detector $C$ is $\psi_A(C) = e^{i(\alpha+\phi)}/\sqrt{2}$ and the amplitude for the photon to arrive at detector $D$ is $\psi_A(D) = e^{i(\alpha-\phi)}/\sqrt{2}$, while if the photon passes through slit $B$, the amplitude for arrival at detector $C$ is $\psi_B(C) = e^{i(\beta+\phi)}/\sqrt{2}$ and the amplitude for arrival at detector $D$ is $\psi_B(D) = -e^{i(\beta-\phi)}/\sqrt{2}$.

*a*) Consider the case where the photon is equally likely to pass through the two slits: the amplitude for arrival at $C$ is $\left(\psi_A(C) + \psi_B(C)\right)/\sqrt{2}$ and the amplitude for arrival at $D$ is $\left(\psi_A(D) + \psi_B(D)\right)/\sqrt{2}$. What is the probability $P(C)$ that $C$ detects the photon and the probability $P(D)$ that $D$ detects the photon. (Recall that the probability is the modulus squared of the amplitude.)

*b*) What are $P(C)$ and $P(D)$ when slit $B$ is covered so that the photon must pass through slit $A$? What if $A$ is covered so the photon must pass through $B$?

*c*) Suppose that both slits are open, but a phase shifter is placed in front of slit $A$, which advances $\alpha$ by $\pi$. What effect does the phase shifter have on $P(C)$ and $P(D)$?