Ph 12b

Homework Assignment No. 2
Due: 5pm, Thursday, 21 January 2010

1. Quantized rotor. A wheel spinning in a plane can be described as a Hamiltonian dynamical system with one degree of freedom: the coordinate is the angular orientation θ taking values in the interval $[0, 2\pi)$, and the conjugate momentum is the angular momentum L. The Hamiltonian H is

$$H = L^2/2I,$$

where I is the moment of inertia.

a) What are the Hamilton equations of motion for this system? Is there a conserved constant of the motion? What is the associated symmetry?

In quantum mechanics, the Hilbert space for this system is the space of square-integrable periodic functions of θ, i.e. functions with the properties

$$\psi(\theta + 2\pi) = \psi(\theta), \quad \int_0^{2\pi} d\theta |\psi(\theta)|^2 < \infty.$$

The angular momentum operator becomes

$$\hat{L} = -i\hbar \frac{d}{d\theta}.$$

b) Find the eigenvalues and normalized eigenfunctions of the operator $\hat{L}$. That is, find all values of λ and functions $\psi_\lambda(\theta)$ such that

$$\hat{L}\psi_\lambda(\theta) = \lambda\psi(\theta), \quad \psi_\lambda(\theta + 2\pi) = \psi_\lambda(\theta), \quad \int_0^{2\pi} d\theta |\psi_\lambda(\theta)|^2 = 1.$$

c) Verify that the eigenfunctions with distinct eigenvalues are mutually orthogonal:

$$\int_0^{2\pi} d\theta \ \psi_\lambda(\theta)^* \psi_{\lambda'}(\theta) = 0 \quad \text{for} \quad \lambda \neq \lambda'.$$

d) What are the eigenvalues and eigenfunctions of the Hamiltonian $\hat{H} = \hat{L}^2/2I$?

e) The expectation value of the angular momentum is

$$\langle \hat{L} \rangle = \int_0^{2\pi} d\theta \; \psi(\theta)^* \hat{L} \psi(\theta).$$

Show that if the wavefunction $\psi(\theta)$ is *real* (*i.e.* $\psi(\theta) = \psi(\theta)^*$), then $\langle \hat{L} \rangle = 0$.

2. Twisted rotor. Now consider a nonstandard way to quantize the spinning wheel — the wavefunction $\psi(\theta)$ is not periodic, but instead "periodic up to a phase":

$$\psi(\theta + 2\pi) = e^{i\alpha} \psi(\theta),$$

where $e^{i\alpha}$ is a fixed complex number with modulus one. For this "twisted rotor," repeat parts (*b*)–(*d*) of Problem 1.

3. More eigenfunctions. For square-integrable functions on the real line, consider the Hermitian operator

$$\hat{H} = -\frac{d^2}{dx^2} + x^2.$$

a) Show that the functions

$$\psi_0(x) = e^{-x^2/2}, \quad \psi_1(x) = x e^{-x^2/2}$$

are eigenfunctions of $\hat{H}$, and find their eigenvalues. Check that $\psi_0(x)$ and $\psi_1(x)$ are orthogonal functions.

b) Find a real value of C such that

$$\psi_2(x) = (x^2 + C) e^{-x^2/2}$$

is an eigenfunction of $\hat{H}$, and find its eigenvalue.

c) Check that ψ_2 is orthogonal to ψ_0 and ψ_1. It's useful to recall that

$$\int_{-\infty}^{\infty} dx \; e^{-x^2} = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} dx \; x^2 e^{-x^2} = \sqrt{\pi}/2.$$

4. The qubit. A *qubit* is a quantum system whose Hilbert space is two dimensional; linear operators acting on a qubit are 2×2 matrices.

a) Show that the most general Hermitian operator acting on a qubit can be expressed as

$$a\hat{I} + b\hat{\sigma}(\theta, \phi)$$

where

$$\hat{\sigma}(\theta, \phi) = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}.$$

Here $\hat{I}$ is the 2×2 identity matrix, a is an arbitrary real number, b is a nonnegative real number, θ is a real number in the interval $[0, \pi]$, and ϕ is a real number in the interval $[0, 2\pi)$.

b) Find the eigenvalues and eigenvectors of the operator $\hat{\sigma}(\theta, \phi)$. It is convenient to express the eigenvectors in terms of $\cos(\theta/2)$, $\sin(\theta/2)$, $e^{i\phi/2}$ and $e^{-i\phi/2}$.

The 2×2 *Pauli spin matrices* are the Hermitian operators

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

c) For both of the eigenvectors found in (*b*), evaluate the expectation values

$$\langle\hat{\sigma}_1\rangle, \quad \langle\hat{\sigma}_2\rangle, \quad \langle\hat{\sigma}_3\rangle.$$

It is convenient to express the expectation values in terms of $\cos\theta$, $\sin\theta$, $\cos\phi$, $\sin\phi$.

d) If the observable $\hat{\sigma}_3$ is measured, the outcome can be either one of its eigenvalues, $+1$ or -1. For both of the eigenvectors found in part (*b*), find the probability $P(+)$ for the $+1$ outcome of a $\hat{\sigma}_3$ measurement and the probability $P(-)$ for the -1 outcome of a $\hat{\sigma}_3$ measurement.