Ph 12b

Homework Assignment No. 4
Due: 5pm, Thursday, 4 February 2010

1. **Weaker decoherence**. In class we discussed the *phase damping* of a qubit that results when the qubit scatters a photon with probability p. The scattered photon is knocked into one of two mutually orthogonal states $\{|0\rangle_E, |1\rangle_E\}$, correlated with the qubit's state, both of which are orthogonal to the state $|\text{un}\rangle_E$ of the unscattered photon. If the initial state of the qubit is $|\psi\rangle_S = a|0\rangle_S + b|1\rangle_S$, then the joint state of the qubit and photon evolves as

$$|\psi\rangle_S \otimes |\text{un}\rangle_E \to \sqrt{1-p}\,|\psi\rangle_S \otimes |\text{un}\rangle_E + \sqrt{p}\,(a|0\rangle_S \otimes |0\rangle_E + b|1\rangle_S \otimes |1\rangle_E)\,. \tag{1}$$

Thus the qubit density operator $\hat{\rho}$ evolves as

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \to \hat{\rho}' = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}.$$

Now consider a different model of decoherence, in which photon scattering does not perfectly resolve the state of the qubit. The scattered photon is knocked to the normalized state $|\gamma\rangle_E$ if the qubit's state is $|0\rangle_S$ and it is knocked to the normalized state $|\eta\rangle_E$ if the photon's state is $|1\rangle_S$; thus eq.(1) is replaced by

$$|\psi\rangle_S \otimes |\text{un}\rangle_E \to \sqrt{1-p}\,|\psi\rangle_S \otimes |\text{un}\rangle_E + \sqrt{p}\,(a|0\rangle_S \otimes |\gamma\rangle_E + b|1\rangle_S \otimes |\eta\rangle_E)\,. \tag{2}$$

Both $|\gamma\rangle_E$ and $|\eta\rangle_E$ are orthogonal to the state $|\text{un}\rangle_E$ of the unscattered photon, but they are not necessarily mutually orthogonal; rather

$${}_E\langle\eta|\gamma\rangle_E = 1 - \epsilon,$$

where ϵ is a real number. Thus for $\epsilon = 1$, the states $|\gamma\rangle_E$ and $|\eta\rangle_E$ are orthogonal, and we recover the model considered previously, while for $\epsilon = 0$, the scattered photon remains uncorrelated with the qubit, and there is no decoherence at all.

Show that eq.(2) implies that the density operator evolves according to

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \to \hat{\rho}' = \begin{pmatrix} \rho_{00} & \lambda\rho_{01} \\ \lambda\rho_{10} & \rho_{11} \end{pmatrix},$$

and express λ in terms of p and ϵ. You may find it convenient to expand $|\gamma\rangle_E$ and $|\eta\rangle_E$ in terms of two orthonormal vectors $|e_0\rangle_E$ and $|e_1\rangle_E$ (which are both orthogonal to $|\text{un}\rangle_E$), so that

$$|\gamma\rangle_E = \gamma_0|e_0\rangle_E + \gamma_1|e_1\rangle_E, \quad |\eta\rangle_E = \eta_0|e_0\rangle_E + \eta_1|e_1\rangle_E.$$

2. Master equation for spontaneous decay. A two-level atom has a ground state $|g\rangle$ with energy E_g and an excited state $|e\rangle$ with energy $E_e = E_g + \hbar\omega$. We may adjust the definition of energy by an additive constant, so that the ground state has zero energy, and the excited state's energy is $\hbar\omega$. Thus the Hamiltonian of the atom is

$$\hat{H} = \hbar\omega|e\rangle\langle e|,$$

and the state vector $|\psi(t)\rangle$ of the atom evolves according to

$$|\psi(t+dt)\rangle = |\psi(t)\rangle - i\omega dt|e\rangle\langle e|\psi\rangle,$$

where dt is an infinitesimal time increment.

a) Recall that a general density operator $\hat{\rho}$ for the atom can be represented as $\hat{\rho} = \sum_a p_a|\psi_a\rangle\langle\psi_a|$, where each $|\psi_a\rangle$ is a normalized state vector and the p_a's are positive real numbers satisfying $\sum p_a = 1$. Show that the time-evolving density operator $\hat{\rho}(t)$ obeys the differential equation

$$\frac{d}{dt}\hat{\rho} = -i\omega|e\rangle\langle e|\hat{\rho} + i\omega\hat{\rho}|e\rangle\langle e|. \tag{3}$$

b) Writing the density operator as the matrix

$$\hat{\rho} = \begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix}$$

in the basis $\{|g\rangle, |e\rangle\}$, express eq.(3) as four separate differential equations for $\rho_{gg}(t) = \langle g|\hat{\rho}|g\rangle$, $\rho_{ge}(t) = \langle g|\hat{\rho}|e\rangle$, $\rho_{eg}(t) = \langle e|\hat{\rho}|g\rangle$, $\rho_{ee}(t) = \langle e|\hat{\rho}|e\rangle$. Solve these equations, finding $\hat{\rho}(t)$ in terms of $\hat{\rho}(0)$.

Now suppose that the excited state of the atom can decay to the ground state by emitting a photon. The rate for the decay process is

Γ. Thus in the time interval $(t, t+dt)$ the joint state of the atom and its environment evolves according to

$$|\psi(t)\rangle \otimes |0\rangle \rightarrow \quad \Big(|g\rangle\langle g|\psi(t)\rangle + \sqrt{1-\Gamma dt}\ |e\rangle\langle e|\psi(t)\rangle\Big) \otimes |0\rangle + \sqrt{\Gamma dt}\ |g\rangle\langle e|\psi(t)\rangle \otimes |1\rangle. \quad (4)$$

Here $|0\rangle$ denotes the state of the environment containing no photon, and $|1\rangle$ denotes the state of the environment containing one photon. (For now we are considering only the evolution due to spontaneous decay, we are ignoring the evolution arising from the Hamiltonian $\hat{H}$.)

c) Show that eq.(4) implies a differential equation satisfied by the density operator

$$\frac{d}{dt}\hat{\rho} = \Gamma|g\rangle\langle e|\hat{\rho}|e\rangle\langle g| - \frac{1}{2}\Gamma|e\rangle\langle e|\hat{\rho} - \frac{1}{2}\Gamma\hat{\rho}|e\rangle\langle e|. \quad (5)$$

Eq.(5) is called the atom's *master equation.*

d) Extract from eq.(5) differential equations satisfied by ρ_{ee}, ρ_{eg}, and ρ_{ge}. Solve these equations, finding $\hat{\rho}(t)$ in terms of $\hat{\rho}(0)$.

e) When we combine eq.(5) describing spontaneous decay with eq.(3) describing the atom's evolution governed by the Schrödinger equation, we obtain a new master equation

$$\frac{d}{dt}\hat{\rho} = \left(-i\omega - \frac{1}{2}\Gamma\right)|e\rangle\langle e|\hat{\rho} + \left(i\omega - \frac{1}{2}\Gamma\right)\hat{\rho}|e\rangle\langle e| + \Gamma|g\rangle\langle e|\hat{\rho}|e\rangle\langle g|. \quad (6)$$

Again, find differential equations satisfied by ρ_{ee}, ρ_{eg}, and ρ_{ge} and solve them, determining $\hat{\rho}(t)$ in terms of $\hat{\rho}(0)$.

3. Diagonalizing the density operator. Suppose that the state of a qubit is prepared by flipping a fair coin, and then preparing the state vector $|\psi_H\rangle$ if the outcome of the coin flip is heads, and preparing the state vector $|\psi_T\rangle$ if the outcome of the coin flip is tails, where

$$|\psi_H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi_T\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

a) Find the density operator $\hat{\rho}$ of the qubit.

b) What are the eigenvalues of $\hat{\rho}$?

c) Find the eigenvectors of $\hat{\rho}$. It is convenient to express your answer in terms of $\cos(\pi/8)$ and $\sin(\pi/8)$.

4. Schmidt decomposition. Consider a composite quantum system AB with Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where

$$\text{dimension}\,(\mathcal{H}_A) = \text{dimension}\,(\mathcal{H}_B) = N.$$

If $\{|e_a\rangle, a = 1, 2, 3, \ldots N\}$ is an orthonormal basis for $\mathcal{H}_A$ and and $\{|f_b\rangle, b = 1, 2, 3, \ldots N\}$ is an orthonormal basis for $\mathcal{H}_B$, then a normalized state vector $|\psi\rangle \in \mathcal{H}_{AB}$ can be expanded as

$$|\psi\rangle = \sum_{a,b=1}^{N} \psi_{ab}\, |e_a\rangle \otimes |f_b\rangle,$$

where

$$\sum_{a,b=1}^{N} |\psi_{ab}|^2 = 1.$$

a) Any $N \times N$ matrix M has a *singular value decomposition* $M = UDV$, where U and V are $N \times N$ unitary matrices, and D is a diagonal real $N \times N$ matrix with nonnegative eigenvalues. Use the singular value decomposition to show that for any given state vector $|\psi\rangle \in \mathcal{H}_{AB}$, one can choose an orthonormal basis $\{|e'_a\rangle, a = 1, 2, 3, \ldots N\}$ for $\mathcal{H}_A$ and an orthonormal basis $\{|f'_b\rangle, b = 1, 2, 3, \ldots N\}$ for $\mathcal{H}_B$ such that $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \sum_{a=1}^{N} \sqrt{p_a}\, |e'_a\rangle \otimes |f'_a\rangle, \tag{7}$$

where the $\{p_a\}$ are nonnegative real numbers such that

$$\sum_{a=1}^{N} p_a = 1.$$

This expression is called the *Schmidt decomposition* of the state vector $|\psi\rangle$.

b) Using the expression for $|\psi\rangle$ in eq.(7), express the density operator $\hat{\rho}_A$ for system A in the basis $\{|e'_a\rangle\}$ and express the density operator $\hat{\rho}_B$ for system B in the basis $\{|f'_a\rangle\}$.

c) What are the eigenvalues of $\hat{\rho}_A$ and of $\hat{\rho}_B$?