Ph 12b

Homework Assignment No. 7
Due: 5pm, Thursday, 4 March 2010

1. Damped harmonic oscillator (15 points).

Let's suppose the oscillations of a quantum harmonic oscillator with circular frequency ω are damped because the oscillator can emit photons with energy $\hbar\omega$. When a photon is emitted, the oscillator makes a transition from the energy eigenstate with energy $E_n = n\hbar\omega$ to the energy eigenstate with energy $E_{n-1} = (n-1)\hbar\omega$, and the photon carries away the lost energy. The probability that a photon is emitted in an infinitesimal time interval dt is Γdt; we say that Γ is the emission rate. Therefore, the coupled evolution of the oscillator and the electromagnetic field for time interval dt can be described as:

$$\begin{aligned}&|\Psi(0)\rangle = |\psi\rangle \otimes |0\rangle \to \\ &|\Psi(dt)\rangle = \sqrt{\Gamma dt}\ \hat{a}|\psi\rangle \otimes |1\rangle + \left(\hat{I} - \frac{1}{2}\Gamma dt\ \hat{a}^\dagger \hat{a}\right)|\psi\rangle \otimes |0\rangle.\end{aligned}$$

Here $|\psi\rangle$ is the initial normalized state vector of the oscillator and $\{|0\rangle, |1\rangle\}$ are orthonormal states of the electromagnetic field; $|0\rangle$ denotes the state in which no photon has been emitted and $|1\rangle$ denotes the state containing one photon. The operator $\hat{a}$ reduces the excitation level of the oscillator by one unit, and the $\hat{a}^\dagger\hat{a}$ factor in the second term is needed to ensure that the evolution is unitary.

a) Check unitarity by verifying that $\langle\Psi(dt)|\Psi(dt)\rangle = 1$, to linear order in the small quantity dt.

Because the states $\{|0\rangle, |1\rangle\}$ of the electromagnetic field are orthogonal, the quantum state of the oscillator may decohere. Summing over these basis states, we see that the initial pure state $|\psi\rangle\langle\psi|$ of the oscillator evolves in time dt as

$$\begin{aligned}|\psi\rangle\langle\psi| \quad &\to \quad \langle 0|\Psi(dt)\rangle\langle\Psi(dt)|0\rangle + \langle 1|\Psi(dt)\rangle\langle\Psi(dt)|1\rangle \\ &= \quad \Gamma dt\ \hat{a}|\psi\rangle\langle\psi|\hat{a}^\dagger + \left(\hat{I} - \frac{1}{2}\Gamma dt\ \hat{a}^\dagger\hat{a}\right)|\psi\rangle\langle\psi|\left(\hat{I} - \frac{1}{2}\Gamma dt\ \hat{a}^\dagger\hat{a}\right);\end{aligned}$$

more generally, the initial (not necessarily pure) density operator $\hat{\rho}$ of the oscillator evolves as

$$\hat{\rho} \to \Gamma dt\ \hat{a}\hat{\rho}\hat{a}^\dagger + \left(\hat{I} - \frac{1}{2}\Gamma dt\ \hat{a}^\dagger\hat{a}\right)\hat{\rho}\left(\hat{I} - \frac{1}{2}\Gamma dt\ \hat{a}^\dagger\hat{a}\right). \quad (1)$$

Now suppose that the initial state of the oscillator is a coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

where α is a complex number. For this problem, we will ignore the usual dynamics of the oscillator that causes α to rotate uniformly in time: $\alpha \to \alpha e^{-i\omega t}$; equivalently, we will assume that the dynamics is described in a "rotating frame" such that the rotation of α is transformed away. We will only be interested in how the states of the oscillator are affected by the damping described by eq.(1).

b) Show that, to linear order in dt,

$$\left(\hat{I} - \frac{1}{2}\Gamma dt\ \hat{a}^\dagger\hat{a}\right)|\alpha\rangle \approx e^{-\Gamma dt|\alpha|^2/2}|\alpha\ e^{-\Gamma dt/2}\rangle. \quad (2)$$

Note that there are two things to check in eq.(2): that the value of α decays with time, and that the normalization of the state decays with time.

c) Verify that, also to linear order in dt,

$$\Gamma dt\ \hat{a}|\alpha\rangle\langle\alpha|\hat{a}^\dagger \approx \Gamma dt|\alpha|^2\ |\alpha\ e^{-\Gamma dt/2}\rangle\langle\alpha\ e^{-\Gamma dt/2}|,$$

and thus show that, to linear order in dt, $|\alpha\rangle\langle\alpha|$ evolves as

$$|\alpha\rangle\langle\alpha| \to |\alpha\ e^{-\Gamma dt/2}\rangle\langle\alpha\ e^{-\Gamma dt/2}|.$$

By considering many consecutive small time increments, argue that, in a finite time t, the initial coherent state evolves as

$$|\alpha\rangle \to |\alpha\ e^{-\Gamma t/2}\rangle.$$

Thus, the state remains a (pure) coherent state at all times, with the value of α decaying exponentially with time. Since the energy stored in the oscillator is proportional to $|\alpha|^2$, which decays like $e^{-\Gamma t}$, we may say that $\Gamma \equiv \Gamma_{\text{damp}}$ is the *damping rate* of the oscillator.

Now consider what happens if the initial state of the oscillator is a superposition of two coherent states:

$$|\psi\rangle = N_{\alpha,\beta}\left(|\alpha\rangle + |\beta\rangle\right).$$

Here $N_{\alpha,\beta}$ is a real nonnegative normalization constant (note that, though the states $|\alpha\rangle$ and $\beta\rangle$ are both normalized, they are not necessarily orthogonal).

d) Evaluate $\langle\beta|\alpha\rangle$, and determine $N_{\alpha,\beta}$.

For example we might choose $\alpha = \xi_0/\sqrt{2}$ and $\beta = -\xi_0/\sqrt{2}$, so that the two superposed coherent states are minimum uncertainty wavepackets (with width $\Delta\xi = 1/\sqrt{2}$) centered at dimensionless positions $\pm\xi_0$. If $|\alpha - \beta| \gg 1$, then the two wavepackets are well separated compared to their width, and we might say that oscillator state $|\psi\rangle$ is "in two places at once." How quickly will such a superposition of two separated wavepackets decohere?

The initial density operator of the oscillator is

$$\hat{\rho} = N_{\alpha,\beta}^2(|\alpha\rangle\langle\alpha| + |\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha| + |\beta\rangle\langle\beta|).$$

We already know from part (c) how the "diagonal" terms $|\alpha\rangle\langle\alpha|$ and $|\beta\rangle\langle\beta|$ evolve, but what about the "off-diagonal" terms $|\alpha\rangle\langle\beta|$ and $|\beta\rangle\langle\alpha|$?

e) Using arguments similar to those used in parts (b) and (c), show that in time t, the operator $|\alpha\rangle\langle\beta|$ evolves as

$$|\alpha\rangle\langle\beta| \to (\text{phase})e^{-\Gamma t|\alpha-\beta|^2/2}|\alpha e^{-\Gamma t/2}\rangle\langle\beta e^{-\Gamma t/2}|,$$

where (phase) denotes a phase factor. Thus the off-diagonal terms decay exponentially with time, at a rate

$$\Gamma_{\text{decohere}} = \frac{1}{2}|\alpha - \beta|^2\ \Gamma_{\text{damp}}$$

proportional to the distance squared $|\alpha - \beta|^2$.

f) Consider an oscillator with mass $m = 1\ g$, circular frequency $\omega = 1\ s^{-1}$ and (*very good*) quality factor $Q \equiv \omega/\Gamma = 10^9$. Thus the damping time is very long: over 30 years. A superposition of minimum uncertainty wavepackets is prepared, centered at positions $x = \pm 1\ cm$. Estimate the decoherence rate. (Wow! For macroscopic objects, decoherence is really *fast*! And here we have ignored the effects of a nonzero temperature in the environment, which would make it even faster.)

2. A narrow well (10 points).

Consider a particle with mass m moving in the potential

$$V = -\frac{\hbar^2 \Delta}{m}\delta(x),$$

where $\delta(x)$ denotes the Dirac δ-function. The potential is attractive for $\Delta > 0$ and repulsive for $\Delta < 0$. The general solution to the time-dependent Schrödinger equation, for energy $E_0 > 0$, has the form

$$\begin{aligned} \varphi(x) &= Ae^{ikx} + Be^{-ikx}, \quad x < 0, \\ \varphi(x) &= Ce^{ikx} + De^{-ikx}, \quad x > 0, \end{aligned}$$

where $k^2 = 2mE_0/\hbar^2$, and we can determine C and D in terms of A and B using matching conditions at the origin. One matching condition is

$$\lim_{\epsilon \to 0} \left(\varphi(x-\epsilon) - \varphi(x+\epsilon)\right) = 0.$$

We obtain another matching condition by integrating the Schrödinger equation over the interval $[-\epsilon, \epsilon]$:

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} dx \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2 \Delta}{m}\delta(x) \right) \varphi(x) \\ = &-\lim_{\epsilon \to 0} \frac{\hbar^2}{2m} \left(\frac{d}{dx}\varphi(\epsilon) - \frac{d}{dx}\varphi(-\epsilon) + 2\Delta\varphi(0) \right) \\ = &\ E_0 \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \varphi(x) = 0; \end{aligned}$$

that is,

$$\lim_{\epsilon \to 0} \left(\frac{d}{dx}\varphi(\epsilon) - \frac{d}{dx}\varphi(-\epsilon) \right) = -2\Delta\varphi(0). \tag{3}$$

a) Solve the matching conditions, finding a 2×2 matrix $M(\alpha)$ expressed in terms of $\alpha = \Delta/k$ such that

$$\begin{pmatrix} C \\ D \end{pmatrix} = M(\alpha) \begin{pmatrix} A \\ B \end{pmatrix}.$$

b) Find the inverse matrix $M^{-1}(\alpha)$ such that

$$\begin{pmatrix} A \\ B \end{pmatrix} = M^{-1}(\alpha) \begin{pmatrix} C \\ D \end{pmatrix}.$$

c) If there is no incoming wave from the left, then $D = 0$. Under this assumption, use the result from (*b*) to find the transmission amplitude C/A and the reflection amplitude B/A. Square these to compute

$$T(\alpha) = \left|\frac{C}{A}\right|^2, \quad R(\alpha) = \left|\frac{B}{A}\right|^2$$

in terms of α, and verify that $R + T = 1$.

d) Find an imaginary value of $k = i\kappa$ such that the transmission amplitude C/A diverges. This divergence signifies the existence of a solution to the Schrödinger equation with $A = 0$ as well as $D = 0$. Show that for $\Delta > 0$ (the case of an attractive potential) this solution is a normalizable bound state solution. (This connection between poles in the transmission amplitude and bound states is actually a general phenomenon.)

3. Two narrow wells (15 points).

Now consider a particle with mass m moving in the potential

$$V = -\frac{\hbar^2\Delta}{m}\delta(x+a) - \frac{\hbar^2\Delta}{m}\delta(x-a);$$

there are two δ-functions, of equal strength, centered at $x = -a$ and at $x = +a$. The general solution has the form

$$\begin{aligned} \varphi(x) &= Ae^{ikx} + Be^{-ikx}, \quad x < -a, \\ \varphi(x) &= Ce^{ikx} + De^{-ikx}, \quad -a < x < a, \\ \varphi(x) &= Ee^{ikx} + Fe^{-ikx}, \quad x > a, \end{aligned}$$

where $k^2 = 2mE_0/\hbar^2$.

a) Solve the matching conditions at $x = -a$ and $x = a$ to find matrices $M(\alpha, a)$ and $N(\alpha, a)$, and their inverses $M^{-1}(\alpha, a)$ and $N^{-1}(\alpha, a)$, such that

$$\begin{pmatrix} A \\ B \end{pmatrix} = M^{-1}(\alpha, a) \begin{pmatrix} C \\ D \end{pmatrix}, \quad \begin{pmatrix} C \\ D \end{pmatrix} = N^{-1}(\alpha, a) \begin{pmatrix} E \\ F \end{pmatrix},$$

where $\alpha = \Delta/k$.

b) Assuming that $F = 0$, find B/E, compute its square $|B|^2/|E^2| = R/T$, and find an expression for $1/T$.

c) For fixed Δ and k, how should a be chosen to maximize or minimize the transmission?