Ph 12b

Homework Assignment No. 8
Due: 5pm, Thursday, 11 March 2010

1. A barrier in a well (10 points).

A free quantum-mechanical particle with mass m moves inside a one-dimensional box with impenetrable walls located at $x = \pm a$. Furthermore, a *repulsive* δ-function barrier sits at the center of the well, so the potential energy function $V(x)$ in between the two impenetrable walls is given by

$$V(x) = \frac{\hbar^2}{m}\Delta\delta(x) ,$$

where $\delta(x)$ denotes the Dirac δ-function and $\Delta > 0$. As explained in Problem 2 last week, this δ-function potential causes the logarithmic derivative of the wave function $\varphi(x)$ to jump discontinuously at the origin:

$$\varphi'(0^+) - \varphi'(0^-) = 2\Delta\varphi(0) .$$

Here $\varphi'(x)$ denotes the first derivative of $\varphi(x)$, and $\varphi'(0^+)$ (respectively $\varphi'(0^-)$) denotes the limit of $\varphi'(x)$ as x approaches zero from positive (negative) values. The sign convention used here for Δ is the opposite of that used last week; now $\Delta > 0$ is the case of a repulsive barrier.

a) For the even energy eigenstates, what is the value of the logarithmic derivative $\varphi'(x)/\varphi(x)$ at $x = 0^+$ and $x = 0^-$?

b) For the even energy eigenstates, derive an equation that determines the wavenumber k implicitly, where $E = \hbar^2k^2/2m$. Express your answer in the form

$$\Delta a = f(ka),$$

where f is a suitable function.

c) Consider the limiting case of an infinitely strong repulsive barrier: $\Delta a \to \infty$. What are values of the energy eigenvalues in this limit, for both even and odd n?

d) Draw rough sketches of the wave functions for the ground state and the first excited state in the limit $\Delta a \to \infty$.

2. Reflectionless potential (15 points).

Consider a particle with mass m moving in the attractive potential

$$V(x) = -\frac{\hbar^2 k_0^2}{m}\,\text{sech}^2(k_0 x),$$

where $\text{sech}(z) = 2\,(e^z + e^{-z})^{-1}$ denotes the hyperbolic secant function.

a) Show that the time-independent Schrödinger equation for this potential can be expressed as

$$\left(-\frac{d^2}{dz^2} - 2\,\text{sech}^2(z)\right)\varphi(z) = \bar{k}^2\varphi(z), \tag{1}$$

where $z = k_0 x$ is a dimensionless position variable, and $\bar{k}^2 k^2/k_0^2 = 2mE/\hbar^2 k_0^2$ is a dimensionless wavenumber.

b) Show that

$$(i\bar{k} - \tanh(z))e^{i\bar{k}z} \tag{2}$$

solves eq.(1).

c) Show that eq.(2) approaches $Ae^{i\bar{k}z}$ as $z \to -\infty$ and approaches $Ce^{i\bar{k}z}$ as $z \to +\infty$, where A and C are constants. What are the values of these constants.

d) What is the transmission amplitude C/A? Show that the transmission probability $T = |C/A|^2$ is one, and that the reflection probability $R = 1 - T$ is zero. Hence, if a wavepacket is incident on this potential from the far left, there is no reflected wave packet at all.

e) Find an imaginary value $\bar{k} = i\bar{\kappa}$ such that $A/C = 0$ and the transmission amplitude thus diverges. For this value of $\bar{\kappa}$, there is a normalizable bound state solution, which decays exponential for both $z \to +\infty$ and $z \to -\infty$.

f) Check that

$$\varphi(z) = \text{sech}(z)$$

solves eq.(1), where $\bar{k}^2 = -\bar{\kappa}^2$ and $\bar{\kappa}$ is the value found in (*e*). This is the bound state solution. What is the corresponding bound state energy?

3. Bound states in a linear potential (15 points).

Consider a particle with mass m moving in the potential

$$V(x) = F|x|$$

where $|x|$ denotes the absolute value function. Thus there is a constant force F directed toward the origin.

a) Show that the time-independent Schrödinger equation for this potential can be expressed in the form (for $x \geq 0$)

$$\left(-\frac{d^2}{dy^2} + y\right) \varphi(y) = \bar{E}\ \varphi(y), \tag{3}$$

where

$$y = \left(\frac{\hbar^2}{2mF}\right)^{-1/3} x, \quad \bar{E} = \left(\frac{\hbar^2 F^2}{2m}\right)^{-1/3} E.$$

Equivalently, we may write

$$\frac{d^2}{dz^2}\ \varphi(z) = z\ \varphi(z)$$

where $z = y - \bar{E}$. The solution to this equation that decays as $z \to +\infty$ is the Airy function $\mathrm{Ai}(z)$.

All real zeros of $\mathrm{Ai}(z)$ and of its first derivative $\mathrm{Ai}'(z)$ occur for $z < 0$. We denote the zeros of $\mathrm{Ai}'(z)$, in order of increasing absolute value, by $a_0, a_2, a_4, \ldots$, and we denote the zeros of $\mathrm{Ai}(z)$ in order of increasing absolute value by $a_1, a_3, a_5, \ldots$. These constants have the numerical values:

$$\begin{aligned} -a_0 &= 1.0188\ldots \\ -a_1 &= 2.3381\ldots \\ -a_2 &= 3.2482\ldots \\ -a_3 &= 4.0879\ldots \\ -a_4 &= 4.8201\ldots \\ -a_5 &= 5.5206\ldots \end{aligned} \tag{4}$$

b) Show that for $n = 0, 1, 2, \ldots$ there is a bound state solution to the Schrödinger equation with n nodes and dimensionless "energy" $\bar{E} = \bar{E}_n = -a_n$.

Using the WKB approximation and the connection formulas, we can derive the Bohr-Sommerfeld criterion:

$$\int_{x_1}^{x_2} dx\; k(x) = \pi \left(n + \frac{1}{2} \right) ,$$

where n is the number of nodes in the bound state wavefunction, E_n is the corresponding energy, x_1 and x_2 are the classical turning points for $E = E_n$, and $k(x)^2 = 2m\left(E_n - V(x)\right)/\hbar^2$. For the harmonic potential, this WKB estimate actually agrees with the exact value of E_n, but in general there are corrections higher order in $1/n$.

c) Apply the WKB criterion to the linear potential, deriving a formula for $\bar{E}_n$. For $n = 0, 1, 2, 3, 4, 5$, compare to the exact result from (*b*). You should find pretty good agreement for all $n \geq 1$. Furthermore, you should find (considering the odd and even values of n separately), that the agreement gets systematically better as n increases.