PH12b 2010 Solutions HW#1

1.

a) $\langle x\rangle = 0 = \langle p\rangle$ imply that $(\Delta x)^2 = \langle x^2\rangle$ and $(\Delta p)^2 = \langle p^2\rangle$. Using this we get

$$\langle E\rangle = \frac{\langle p^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2 .$$

Now, from the uncertainty relation we know that $\Delta p \geq \hbar/2\Delta x$, therefore

$$\langle E\rangle \geq \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2 .$$

b)

$$\left.\frac{d\langle E\rangle}{d\Delta x}\right|_{\Delta x_m} = \frac{-2\hbar^2}{8m(\Delta x_m)^3} + m\omega^2(\Delta x_m) = 0,$$

solving for Δx_m we get

$$\Delta x_m = \sqrt{\frac{\hbar}{2m\omega}}.$$

This imply that

$$\langle E\rangle \geq \frac{\hbar\omega}{2}.$$

Then, the value of the energy lower bound for any quantum state is the same as the ground state energy of the one-dimensional harmonic oscillator.

2.

a) First we define

$$\begin{aligned}\overline{x_0} &\equiv x_0 - \langle x_0\rangle ,\\ \overline{p_0} &\equiv p_0 - \langle p_0\rangle .\end{aligned}$$

Because the position and momentum of the particle are "uncorrelated" then

$$\langle\overline{x_0 p_0}\rangle + \langle\overline{p_0 x_0}\rangle = 0.$$

Notice that x_0 and p_0 are operators that do not commute, therefore $\langle x_0 p_0\rangle \neq \langle p_0 x_0\rangle$.

We know that $x_t = x_0 + p_0 t/m$ then

$$\begin{aligned}(\Delta x_t)^2 &= \left\langle (x_t - \langle x_t\rangle)^2\right\rangle\\ &= \left\langle\left(\overline{x_0} + \frac{\overline{p_0}t}{m}\right)^2\right\rangle\\ &= (\Delta x_0)^2 + \frac{t^2}{m^2}(\Delta p_0)^2 + \frac{t}{m}\left[\langle\overline{x_0 p_0}\rangle + \langle\overline{p_0 x_0}\rangle\right]\\ &= (\Delta x_0)^2 + \frac{t^2}{m^2}(\Delta p_0)^2 ,\end{aligned}$$

where we used $\langle\overline{x_0 p_0}\rangle + \langle\overline{p_0 x_0}\rangle = 0$. From the uncertainty principle we know that $\Delta p_0 \geq \hbar/2\Delta x_0$, so finally we get

$$(\Delta x_t)^2 \geq (\Delta x_0)^2 + \frac{\hbar^2 t^2}{4m^2(\Delta x_0)^2}.$$

b) Because

$$\Delta x_0 \Delta x_t \geq \sqrt{(\Delta x_0)^4 + \frac{\hbar^2 t^2}{4m^2}},$$

it is obvious that the lower bound is reach when $\Delta x_0 = 0$ and therefore

$$\Delta x_0 \Delta x_t \geq \frac{\hbar t}{2m}.$$

c) From b) we have that

$$\textit{standard quantum limit} = \sqrt{\frac{\hbar t}{2m}} = \sqrt{\frac{(10^{-34} m^2 kg/s)\,(10^{-2} s)}{2\,(10kg)}} \approx 2 \times 10^{-19} m.$$

The size of a proton is around 1fm=$10^{-15}m$, then the *standard quantum limit* is four orders of magnitude smaller.

3.

a) The Hamilton's equations are

$$\dot{x} = \frac{\partial x}{\partial t} = \frac{\partial H}{\partial p}, \qquad \dot{p} = \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial x}.$$

The Hamiltonian of a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2.$$

Therefore the Hamilton's equations of motion for this system are

$$\dot{p} = -m\omega^2 x, \qquad \dot{x} = p/m.$$

b) See Fig.1. c) The equation for the orbits are

$$1 = \frac{p^2}{2mE} + \frac{m\omega^2}{2E} x^2 = \frac{p^2}{a^2} + \frac{x^2}{b^2},$$

then the axis of the ellipse are $a = \sqrt{2mE}$, $b = \sqrt{2E/m\omega^2}$ (or vice versa). Then

$$J(E) = A = \pi ab = \frac{2\pi E}{\omega},$$

d)

$$T = \frac{\partial J}{\partial E} = \frac{2\pi}{\omega},$$

as expected.

e) By the requirement that the action J is an integer multiple of Planck's constant h we get

$$J(E_n) = \frac{2\pi E_n}{\omega} = nh, \qquad n = 0, 1, 2, 3, \ldots$$

Solving for E_n gives the energy levels of the harmonic oscillator

$$E_n = \left(\frac{h}{2\pi}\right) n\omega, \qquad n = 0, 1, 2, 3, \ldots$$

4.

The Poisson bracket $[A, B]$ of A and B is defined as

$$[A, B] = \sum_{a=1}^{N} \left(\frac{\partial A}{\partial q_a} \frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial q_a} \frac{\partial A}{\partial p_a} \right).$$

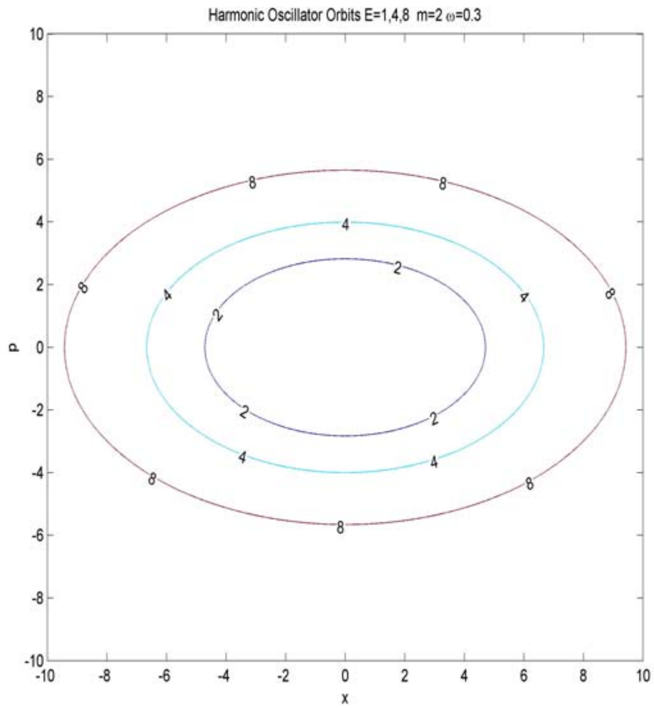

Figure 1: Orbits for several different values of the energy. The direction of the flow along the orbit is CLOCKWISE.

The Hamilton's equations are

$$\frac{\partial q_a}{\partial t} = \frac{\partial H}{\partial p_a}, \qquad \frac{\partial p_a}{\partial t} = -\frac{\partial H}{\partial q_a},$$

where H is the Hamiltonian.

a) Show that

$$\frac{dA}{dt} = [A, H],$$

where $A(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N)$.

Proof

$$\begin{aligned} lhs &= \frac{dA}{dt} = \sum_{a=1}^{N} \left(\frac{\partial A}{\partial q_a} \frac{\partial q_a}{\partial t} + \frac{\partial A}{\partial p_a} \frac{\partial p_a}{\partial t} \right), \\ &= \sum_{a=1}^{N} \left(\frac{\partial A}{\partial q_a} \frac{\partial H}{\partial p_a} - \frac{\partial A}{\partial p_a} \frac{\partial H}{\partial q_a} \right), \\ &= [A, H] = rhs. \end{aligned}$$

b) Show that $[A, B] = 0$ if $B = B(A)$.

Proof

$$\begin{aligned} [A, B] &= \sum_{a=1}^{N} \left(\frac{\partial A}{\partial q_a} \frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial q_a} \frac{\partial A}{\partial p_a} \right), \\ &= \sum_{a=1}^{N} \left(\frac{\partial A}{\partial q_a} \frac{\partial B}{\partial A} \frac{\partial A}{\partial p_q} - \frac{\partial B}{\partial A} \frac{\partial A}{\partial q_a} \frac{\partial A}{\partial p_a} \right), \\ &= 0, \end{aligned}$$

where we used the chain rule i.e. $\partial B/\partial p_a = (\partial B/\partial A)(\partial A/\partial p_a)$.

c) If $\partial H/\partial t = 0$ by a), assuming $A = H$, we have

$$\frac{dH}{dt} = [H, H],$$

now, by b), $[H, H] = 0$, therefore

$$\frac{dH}{dt} = 0.$$

d)

$$[q_a, q_b] = 0,$$

because $\partial q_a/\partial p_b = 0$.

$$[p_a, p_b] = 0,$$

because $\partial p_a/\partial q_b = 0$.

Now,

$$\frac{\partial q_a}{\partial q_b} = \delta_{ab} = \begin{cases} 0 & a \neq b \\ 1 & a = b \end{cases},$$

and

$$\frac{\partial p_a}{\partial p_b} = \delta_{ab} = \begin{cases} 0 & a \neq b \\ 1 & a = b \end{cases},$$

then

$$\begin{aligned} [q_a, p_b] &= \sum_{c=1}^{N} \left(\frac{\partial q_a}{\partial q_c} \frac{\partial p_b}{\partial p_c} - \frac{\partial p_b}{\partial q_c} \frac{\partial q_a}{\partial p_c} \right), \\ &= \sum_{c=1}^{N} \left(\frac{\partial q_a}{\partial q_c} \frac{\partial p_b}{\partial p_c} \right) = \sum_{c=1}^{N} \delta_{ac} \delta_{bc} = \delta_{ab}, \end{aligned}$$

then

$$[q_a, p_b] = \delta_{ab}.$$

5.

a) The amplitudes are

$$\begin{aligned} \psi_C &= (\psi_A(C) + \psi_B(C))/\sqrt{2} = \frac{1}{2} e^{i\phi} \left(e^{i\alpha} + e^{i\beta} \right), \\ \psi_D &= (\psi_A(D) + \psi_B(D))/\sqrt{2} = \frac{1}{2} e^{-i\phi} \left(e^{i\alpha} - e^{i\beta} \right). \end{aligned}$$

Then the probabilities are

$$\begin{aligned} P(C) &= |\psi_C|^2 = \cos\left(\frac{\alpha - \beta}{2} \right)^2, \\ P(D) &= |\psi_D|^2 = \sin\left(\frac{\alpha - \beta}{2} \right)^2. \end{aligned}$$

Notice that $P(C) + P(D) = 1$ as expected.

b) If the slit B is covered then

$$\psi_C = \psi_A(C) = \tfrac{1}{\sqrt{2}} e^{i\phi} e^{i\alpha}, \qquad \psi_D = \psi_A(D) = \tfrac{1}{\sqrt{2}} e^{-i\phi} e^{i\alpha},$$
$$P(C) = |\psi_C|^2 = 1/2, \qquad P(C) = |\psi_D|^2 = 1/2.$$

If the slit A is covered then

$$\psi_C = \psi_B(C) = \tfrac{1}{\sqrt{2}} e^{i\phi} e^{i\beta}, \qquad \psi_D = \psi_A(D) = \tfrac{-1}{\sqrt{2}} e^{-i\phi} e^{i\beta},$$
$$P(C) = |\psi_C|^2 = 1/2, \qquad P(C) = |\psi_D|^2 = 1/2.$$

c) α just appear in ψ_A, then we can use $P(C)$ and $P(D)$ of a) in the following way

$$\begin{aligned} P(C)_{New} &= P(C)|_{\alpha\to\alpha+\pi} = \cos\left(\frac{\alpha-\beta}{2} + \frac{\pi}{2}\right)^2 = \sin\left(\frac{\alpha-\beta}{2}\right)^2 = P(D), \\ P(D)_{New} &= P(D)|_{\alpha\to\alpha+\pi} = \sin\left(\frac{\alpha-\beta}{2} + \frac{\pi}{2}\right)^2 = \cos\left(\frac{\alpha-\beta}{2}\right)^2 = P(C). \end{aligned}$$