Homework 2 Solutions

Ph 12b Winter 2010

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1. Quantized Rotor

a. We know that L is the conjugate momentum of θ. From equation 1.14 in Liboff we can evaluate the equations of motion to see that

$$\frac{\partial H}{\partial L} = \frac{L}{I} = \dot{\theta}.$$

Since θ is missing from the Hamiltonian, it is a cyclic variable and thus L is a constant of the motion. As a result, the system is radially symmetric.

b. To find the eigenvalues and eigenfunctions of $\hat{L}$, we need to consider solutions to the eigenvalue equation

$$-i\hbar\frac{d}{d\theta}\psi = \lambda\psi.$$

Using the solution in section 3.1 of Liboff as a guide, we can see that $\psi = e^{im\theta}$ satisfies the differential equation above and gives us eigenvalues of $\lambda = m\hbar$. For periodicity, we note that

$$e^{im(\theta+2\pi)} = e^{im\theta} \Rightarrow e^{m2\pi} = 1 \text{ and } m \in \mathbb{Z}.$$

Normalization can be achieved by evaluating

$$1 = \int_0^{2\pi} |\psi_\lambda(\theta)|^2 \, d\theta = \int_0^{2\pi} e^{-im\theta} e^{im\theta} \, d\theta = 2\pi \Rightarrow \psi_\lambda = \frac{1}{\sqrt{2\pi}} e^{im\theta}.$$

c. Let $\lambda = m\hbar$ and $\lambda' = n\hbar$, where $m \neq n$ and m, n are integers. Then

$$\int_0^{2\pi} \psi_\lambda^*(\theta)\psi_{\lambda'}(\theta) \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta} e^{in\theta} = \frac{1}{2\pi}\frac{e^{i(n-m)\theta}}{i(n-m)}\bigg|_0^{2\pi} = 0,$$

as required.

d. We need to find the solutions to

$$-\frac{\hbar^2}{2I}\frac{d^2}{d\theta^2}\psi = \lambda\psi.$$

From Liboff, we see that $\psi = e^{im\theta}$ satisfies the differential equation and gives us eigenvalues of $\lambda = \hbar^2 m^2/2I$. The periodicity condition implies that $m \in \mathbb{Z}$, and the normalized eigenfunctions are given by

$$\frac{1}{\sqrt{2\pi}} e^{im\theta}.$$

e.

$$\langle \hat{L} \rangle = \int_0^{2\pi} \psi^*(\theta)\hat{L}\psi(\theta) \, d\theta = -i\hbar \int_0^{2\pi} \psi(\theta)\frac{d}{d\theta}\psi(\theta) \, d\theta$$

Integrating by parts with $u = \psi$ and $dv = d\psi/d\theta$, we get

$$\langle \hat{L} \rangle = -i\hbar \left[\psi^2(\theta) \Big|_0^{2\pi} - \int_0^{2\pi} \psi(\theta) \frac{d\psi}{d\theta}\, d\theta \right] = -i\hbar \psi^2(\theta) \Big|_0^{2\pi} - \langle \hat{L} \rangle \Rightarrow \langle \hat{L} \rangle = -\frac{i\hbar}{2} \psi^2(\theta) \Big|_0^{2\pi}$$

Because of the periodicity condition $\psi(2\pi) = \psi(0)$, $\langle \hat{L} \rangle = 0$.

2. Twisted Rotor

a. Our analysis is the same as in 1b) with the exclusion of the periodicity condition. We instead have

$$e^{im(\theta+2\pi)} = e^{i\alpha} e^{im\theta} \Rightarrow e^{im2\pi} = e^{i\alpha} \Rightarrow m = \frac{\alpha}{2\pi} + n$$

and thus the eigenvalues are $\lambda = \left(\frac{\alpha}{2\pi} + n \right) \hbar$, where $n \in \mathbb{Z}$. Normalizing will produce eigenfunctions similar to the ones above:

$$\psi_\lambda = \frac{1}{\sqrt{2\pi}} e^{i(\alpha/2\pi + n)\theta}.$$

b. Let $\lambda = (\alpha/2\pi + m)\, \hbar$ and $\lambda' = (\alpha/2\pi + n)\, \hbar$, where $m \neq n$ and m, n are integers. Then

$$\int_0^{2\pi} \psi_\lambda(\theta)^* \psi_{\lambda'}(\theta)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} e^{in\theta} = \frac{1}{2\pi} \frac{e^{i(n-m)\theta}}{i(n-m)} \Big|_0^{2\pi} = 0,$$

as required.

c. The eigenfunctions are the same as those derived in part 2a), and the eigenvalues are then

$$\lambda = \frac{\hbar^2 m^2}{2I} = \frac{\hbar^2}{2I} \left(\frac{\alpha}{2\pi} + n \right)^2, \text{ where } n \in \mathbb{Z}.$$

3. More Eigenfunctions

a. To show that ψ_0 and ψ_1 are eigenfunctions of $\hat{H}$, we simply apply the operator to the functions:

$$\hat{H}\psi_0 = \left(-\frac{d^2}{dx^2} + x^2 \right) \psi_0 = e^{-x^2/2} - x^2 e^{-x^2/2} + x^2 e^{-x^2/2} = e^{-x^2/2} = \psi_0,\ \lambda = 1$$

$$\hat{H}\psi_1 = \left(-\frac{d^2}{dx^2} + x^2 \right) \psi_1 = 3xe^{-x^2/2} - x^3 e^{-x^2/2} + x^3 e^{-x^2/2} = 3e^{-x^2/2} = 3\psi_1,\ \lambda = 3$$

Orthogonality can be shown by evaluating

$$\int_{-\infty}^{\infty} \psi_0^*(x) \psi_1(x) dx = \int_{-\infty}^{\infty} e^{-x^2/2} x e^{-x^2/2} dx = \int_{-\infty}^{\infty} x e^{-x^2} dx$$

It is easy to see that since the integrand is an odd function, the integral is zero. Alternatively, one can do a u-substitution with $u = x^2$ and easily show that the integral is zero.

b. Apply ψ_2 to $\hat{H}$:

$$\hat{H}\psi_2 = \left(-\frac{d^2}{dx^2} + x^2 \right) \psi_2 = -\left(x^4 + x^2(C-5) - C + 2 \right) e^{-x^2/2} + x^2 \left(x^2 + C \right) e^{-x^2/2}$$

$$= \left(5x^2 + C - 2 \right) e^{-x^2/2} = \lambda \left(x^2 + C \right) e^{-x^2/2}$$

Let us assume that $\lambda = 5$; the value of C must then be

$$5x^2 + C - 2 = 5(x^2 + C) \Rightarrow C = -\frac{1}{2}.$$

c.

$$\int_{-\infty}^{\infty} \psi_0^*(x)\psi_2(x)dx = \int_{-\infty}^{\infty} e^{-x^2/2}(x^2 - 1/2)e^{-x^2/2}dx = \int_{-\infty}^{\infty} x^2 e^{-x^2}dx - \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}dx = 0 \text{ (see hint)}$$

$$\int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)dx = \int_{-\infty}^{\infty} xe^{-x^2/2}(x^2 - 1/2)e^{-x^2/2}dx = \underbrace{\int_{-\infty}^{\infty} x^3 e^{-x^2}dx}_{=0 \text{ since odd}} - \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} xe^{-x^2}dx}_{=0 \text{ from above}} = 0$$

4. The Qubit

a. We need to show that the operator, for appropriate values of a, b, θ and ϕ, can take a basis vector of the two dimensional Hilbert space and transform it into any other vector. As such, we need to show that the operator, for an appropriate choice of constants, can represent any 2×2 Hermitian matrix. Let us represent the operator as follows:

$$\begin{pmatrix} h & i \\ j & k \end{pmatrix} = \begin{pmatrix} h^* & j^* \\ i^* & k* \end{pmatrix} \text{ (Hermitian property)}$$

From this, we can see that since $h = h^*$ and $k = k^*$, they must be real. We can then express h and k as

$$h = a + c$$
$$k = a - c$$

where a and c are real. We can see that $c = b\cos\theta$, and as a result $a = (h + k)/2$. The constraint $i = j^*$ suggests that we can write the complex number j in the form $de^{-i\phi}$, where $\phi = \arg\{e\}$ and d is a real number that can be represented by $b\sin\theta$, where $b > 0$ represents the radius in polar coordinates. Solving, we get that

$$a = \tfrac{h+k}{2} \quad b = \tfrac{|j|}{\sin\theta} \quad \theta = \tan^{-1}\tfrac{2|j|}{h-k} \quad \phi = \arg\{j\}$$

and thus the operator represents the most general Hermitian operator.

b. We need to solve $\det[\hat{\sigma} - \lambda I] = 0$ to find the eigenvalues:

$$\begin{vmatrix} \cos\theta - \lambda & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta - \lambda \end{vmatrix} = -\cos^2\theta + \lambda^2 - \sin^2\theta = 0, \Rightarrow \lambda = \pm 1$$

Next we find the eigenvectors:

$$\begin{pmatrix} \cos\theta - 1 & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} \eta_1(\cos\theta - 1) + \eta_2 e^{-i\phi}\sin\theta = 0 \\ \eta_1 e^{i\phi}\sin\theta - \eta_2(\cos\theta + 1) = 0 \end{cases}$$

$$\Rightarrow \psi_+ = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) \\ e^{i\phi/2}\sin(\theta/2) \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta + 1 & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta + 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0 \Rightarrow \psi_- = \begin{pmatrix} e^{-i\phi/2}\sin(\theta/2) \\ -e^{i\phi/2}\cos(\theta/2) \end{pmatrix}$$

c. We need to evaluate $\langle\psi|\hat{\sigma}|\psi\rangle$ for all the required expectations. Explicit calculations are only performed for the first expectation. Note: unnormalized eigenvectors result in incorrect expectations. Make sure the eigenvectors used are normed.

$$\langle\psi_+|\hat{\sigma}_1|\psi_1\rangle = \begin{pmatrix} e^{i\phi/2}\cos\frac{\theta}{2} & e^{-i\phi/2}\sin\frac{\theta}{2}\end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} = \cos\phi\sin\theta$$

$$\langle\psi_+|\hat{\sigma}_2|\psi_1\rangle = \sin\phi\sin\theta \qquad \langle\psi_+|\hat{\sigma}_3|\psi_1\rangle = \cos\theta$$

$$\langle\psi_-|\hat{\sigma}_1|\psi_-\rangle = -\cos\phi\sin\theta \quad \langle\psi_-|\hat{\sigma}_2|\psi_-\rangle = -\sin\phi\sin\theta \quad \langle\psi_-|\hat{\sigma}_3|\psi_-\rangle = -\cos\theta$$

d. We need to find the projection of ψ_+ and ψ_- on the eigenvectors of $\hat{\sigma}_3$, which are convieniently $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$: (work is provided only for the first example)

$$P(+) = \langle \text{pos}|\psi_+\rangle = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) \\ e^{i\phi/2}\sin(\theta/2) \end{pmatrix} \right|^2 = \cos^2\left(\frac{\theta}{2}\right)$$

$$P(+) = \langle \text{pos}|\psi_-\rangle = \sin^2\left(\frac{\theta}{2}\right)$$

$$P(-) = \langle \text{neg}|\psi_+\rangle = \sin^2\left(\frac{\theta}{2}\right)$$

$$P(-) = \langle \text{neg}|\psi_-\rangle = \cos^2\left(\frac{\theta}{2}\right)$$