PH12b 2010 Solutions HW#5

1.

a) We solve the differential equation in the following way

$$\left(-i\frac{d}{dx} - i\gamma x - \lambda\right)\psi(x) = 0,$$

$$\begin{aligned}
&\Rightarrow \quad \frac{d}{dx}\psi(x) = (-\gamma x + i\lambda)\psi(x),\\
&\Rightarrow \quad \frac{d\psi(x)}{\psi(x)} = (-\gamma x + i\lambda)\,dx,\\
&\Rightarrow \quad \log\psi(x) = -\frac{1}{2}\gamma x^2 + i\lambda x + c,\\
&\Rightarrow \quad \psi(x) = C\exp\left(-\frac{1}{2}\gamma x^2 + i\lambda x\right).
\end{aligned}$$

where c, C, are constants.

b) We can write the wavefunction in the following way

$$\begin{aligned}
\psi(x) &= C\exp\left(-\frac{1}{2}\gamma x^2 + i\,\mathrm{Re}(\lambda)\,x - \mathrm{Im}(\lambda)\,x\right),\\
&= C\exp\left[-\frac{1}{2}\gamma\left(x + \frac{\mathrm{Im}(\lambda)}{\gamma}\right)^2 + i\,\mathrm{Re}(\lambda)\,x + \frac{\mathrm{Im}(\lambda)^2}{2\gamma}\right],\\
&= C'\exp\left[-\frac{1}{2}\gamma\left(x + \frac{\mathrm{Im}(\lambda)}{\gamma}\right)^2 + i\,\mathrm{Re}(\lambda)\,x\right],
\end{aligned}$$

where C' is another constant. Then, because $|\psi(x)|^2 \propto \exp\left[-\gamma\left(x + \mathrm{Im}(\lambda)/\gamma\right)^2\right]$ and γ is a real number, it is easy to see that the wave function is normalizable if

$$\gamma > 0.$$

c) Lets calculate $\langle\hat{x}\rangle$ and $\left\langle\hat{k}\right\rangle$.

$$\begin{aligned}
\langle\hat{x}\rangle &= \int_{-\infty}^{\infty} x\,|\psi(x)|^2\,dx = \int_{-\infty}^{\infty}(x - \mathrm{Im}(\lambda)/\gamma)\,|\psi(x - \mathrm{Im}(\lambda)/\gamma)|^2\,dx,\\
&= \int_{-\infty}^{\infty} x\,|\psi(x - \mathrm{Im}(\lambda)/\gamma)|^2\,dx - \mathrm{Im}(\lambda)/\gamma\int_{-\infty}^{\infty}|\psi(x)|^2\,dx,\\
&= 0 - \mathrm{Im}(\lambda)/\gamma,\\
&= -\mathrm{Im}(\lambda)/\gamma.
\end{aligned}$$

The first integral in the second line vanished trivially by parity since the integrand is an odd function. Now,

$$\begin{aligned}
\left\langle\hat{k}\right\rangle &= \int_{-\infty}^{\infty}\psi^*(x)\left(-i\frac{d}{dx}\right)\psi(x)\,dx,\\
&= -\gamma\int_{-\infty}^{\infty} x\,|\psi(x - \mathrm{Im}(\lambda)/\gamma)|^2\,dx + \mathrm{Re}(\lambda)\int_{-\infty}^{\infty}|\psi(x)|^2\,dx,\\
&= 0 + \mathrm{Re}(\lambda).\\
&= \mathrm{Re}(\lambda).
\end{aligned}$$

Therefore, the expectation value of the position or in this case the center of the wavepacket is given by $\langle x \rangle = -\operatorname{Im}(\lambda)/\gamma$, while the expectation value of the wave-number operator or in this case the center of the wavepacket in momentum space is $\left\langle \hat{k} \right\rangle = \operatorname{Re}(\lambda)$.

2.

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Substituting $\gamma = \sin\theta$, $0 \le \theta \le \pi/2$ in Eq. (4) we get

$$\begin{pmatrix} -i\sin\theta - \lambda & 1 \\ 1 & i\sin\theta - \lambda \end{pmatrix} |\psi\rangle = 0,$$

In order to have a solution

$$\det \begin{pmatrix} -i\sin\theta - \lambda & 1 \\ 1 & i\sin\theta - \lambda \end{pmatrix} = 0,$$

$$\begin{aligned} &\Rightarrow \quad \left(\sin^2\theta + \lambda^2\right) - 1 = 0, \\ &\Rightarrow \quad \lambda^2 = 1 - \sin^2\theta, \\ &\Rightarrow \quad \lambda = \cos\theta. \end{aligned}$$

b) Consider $|\psi\rangle = \binom{a}{b}$ then

$$\begin{pmatrix} -e^{i\theta} & 1 \\ 1 & -e^{-i\theta} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0,$$

$$\begin{aligned} &\Rightarrow \quad -ae^{i\theta} + b = 0, \\ &\Rightarrow \quad b = ae^{i\theta}. \end{aligned}$$

We still need to normalize $|\psi\rangle$ i.e $\langle \psi | \psi \rangle = 1$, after this we get

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix}.$$

Remember that $|\psi\rangle \sim e^{i\alpha} |\psi\rangle$ where $\alpha \in \Re$.

c)

$$\langle\psi| \hat{\sigma}_1 |\psi\rangle = \frac{1}{2} \begin{pmatrix} e^{i\theta/2} & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix} = \cos\theta,$$

$$\langle\psi| \hat{\sigma}_3 |\psi\rangle = \frac{1}{2} \begin{pmatrix} e^{i\theta/2} & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix} = 0,$$

$$\langle\psi| \hat{\sigma}_1^2 |\psi\rangle = \langle\psi| I |\psi\rangle = \langle \psi | \psi \rangle = 1,$$

$$\langle\psi| \hat{\sigma}_3^2 |\psi\rangle = \langle\psi| I |\psi\rangle = \langle \psi | \psi \rangle = 1,$$

then

$$\begin{aligned} \Delta\hat{\sigma}_1 &= \sqrt{1 - \cos^2\theta} = \sin\theta, \\ \Delta\hat{\sigma}_3 &= 1. \\ \Delta\hat{\sigma}_1 \Delta\hat{\sigma}_3 &= \sin\theta \end{aligned}$$

Finally,

$$\langle\psi| [\hat{\sigma}_1, \hat{\sigma}_3] |\psi\rangle = -2i \langle\psi| \hat{\sigma}_2 |\psi\rangle = -2i\sin\theta,$$

$$\Rightarrow \frac{1}{2} \left|\langle\psi| [\hat{\sigma}_1, \hat{\sigma}_3] |\psi\rangle\right| = \sin\theta,$$

this proves that Eq. (3) is satisfied.

d) We can solve the case $\gamma = 1/\sin\theta$ in the same way we solve a) and b). Here we are going to use another method.

We want to solve

$$(\hat{\sigma}_3 - i\sin\theta\hat{\sigma}_1 - \lambda)\left|\phi\right\rangle = 0,$$

however we already have the solution for

$$(\hat{\sigma}_1 - i\sin\theta\hat{\sigma}_3 - \lambda)\left|\psi\right\rangle = 0.$$

Then if we can make the first equation to look like the second one we are done. To do this first notice that we can write the first equation as

$$\begin{aligned} S\left(\hat{\sigma}_3 - i\sin\hat{\sigma}_1 - \lambda\right)S^T S\left|\phi\right\rangle &= 0, \\ \left(S\hat{\sigma}_3 S^T - i\sin S\hat{\sigma}_1 S^T - \lambda\right)\left|\psi\right\rangle &= 0, \end{aligned}$$

where S is an orthogonal matrix i.e. $S^T S = 1$, and $\left|\psi\right\rangle = S\left|\phi\right\rangle$.

Then we only need to find an orthogonal matrix that satisfy

$$\begin{aligned} S\hat{\sigma}_3 S^T &= \hat{\sigma}_1, \\ S\hat{\sigma}_1 S^T &= \hat{\sigma}_3, \end{aligned}$$

to make the first equation to look like the second one. It is easy to check that the matrix that has these properties is

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then, from the results of a) and b) we see that λ is still

$$\lambda = \cos\theta,$$

and that

$$\begin{aligned} \left|\phi\right\rangle &= S^T\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix}, \\ &= \begin{pmatrix} \cos\theta/2 \\ i\sin\theta/2 \end{pmatrix}. \end{aligned}$$

Finally,

$$\begin{aligned} \left\langle\phi\right|\hat{\sigma}_1\left|\phi\right\rangle &= \left\langle\psi\right|S^t\hat{\sigma}_1 S\left|\psi\right\rangle = \left\langle\psi\right|\hat{\sigma}_3\left|\psi\right\rangle = 0, \\ \left\langle\phi\right|\hat{\sigma}_3\left|\phi\right\rangle &= \left\langle\psi\right|S^t\hat{\sigma}_3 S\left|\psi\right\rangle = \left\langle\psi\right|\hat{\sigma}_1\left|\psi\right\rangle = \cos\theta, \end{aligned}$$

$$\left\langle\phi\right|\hat{\sigma}_1^2\left|\phi\right\rangle = 1,$$
$$\left\langle\phi\right|\hat{\sigma}_3^2\left|\phi\right\rangle = 1,$$

then

$$\begin{aligned} \Delta\hat{\sigma}_3 &= \sqrt{1-\cos^2\theta} = \sin\theta, \\ \Delta\hat{\sigma}_1 &= 1. \\ \Delta\hat{\sigma}_1\Delta\hat{\sigma}_3 &= \sin\theta \end{aligned}$$

Finally,

$$\left\langle\phi\right|[\hat{\sigma}_1, \hat{\sigma}_3]\left|\phi\right\rangle = \left\langle\psi\right|[\hat{\sigma}_3, \hat{\sigma}_1]\left|\psi\right\rangle = 2i\sin\theta,$$
$$\Rightarrow \frac{1}{2}\left|\left\langle\phi\right|[\hat{\sigma}_1, \hat{\sigma}_3]\left|\phi\right\rangle\right| = \sin\theta,$$

proving that Eq. (3) is satisfied.

3.

The following general integral of a Gaussian function is useful

$$\int_{-\infty}^{\infty} Ae^{-Bx^2+Cx+F} = A\sqrt{\frac{\pi}{B}}e^{C^2/4B+F}.$$

a) The wavefunctions are

$$\begin{aligned}\psi(x) &= \frac{1}{(2\pi a^2)^{1/4}}e^{-(x-x_0)^2/4a^2},\\ \varphi(x) &= \frac{1}{(2\pi b^2)^{1/4}}e^{-y^2/4b^2}.\end{aligned}$$

Then

$$\begin{aligned}p(y) &= \int_{-\infty}^{\infty} dx\,|\psi(x)|^2\,|\varphi(y-x)|^2,\\
&= \frac{1}{2\pi ab}\int_{-\infty}^{\infty} dx e^{-(x-x_0)^2/2a^2}e^{-y^2/2b^2},\\
&= \frac{1}{2\pi ab}\int_{-\infty}^{\infty} dx\exp\left[-(x-x_0)^2/2a^2-(y-x)^2/2b^2\right],\\
&= \frac{1}{2\pi ab}\int_{-\infty}^{\infty} dx'\exp\left[-x'^2/2a^2-[x'-(y-x_0)]^2/2b^2\right], \qquad x'=x-x_0,\\
&= \frac{1}{2\pi ab}\int_{-\infty}^{\infty} dx'\exp\left[-x'^2\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{(y-x_0)^2}{b^2}x-\frac{(y-x_0)^2}{2b^2}\right],\\
&= \sqrt{\frac{1}{2\pi(a^2+b^2)}}\exp\left[-\frac{(y-x_0)^2}{2b^2}\right]\exp\left[\frac{(y-x_0)^2}{2b^2}\frac{a^2}{(a^2+b^2)}\right],\\
&= \sqrt{\frac{1}{2\pi(a^2+b^2)}}\exp\left[-\frac{1}{2}\frac{(y-x_0)^2}{(a^2+b^2)}\right].\end{aligned}$$

Then

$$p(y) = \sqrt{\frac{1}{2\pi(a^2+b^2)}}\exp\left[-\frac{1}{2}\frac{(y-x_0)^2}{(a^2+b^2)}\right],$$

which is just a Gaussian with variance $\sigma^2 = \left(a^2+b^2\right)$ and mean $\mu = x_0$, then

$$\begin{aligned}\langle y\rangle &= x_0,\\ (\Delta y)^2 &= \left\langle (y-\langle y\rangle)^2\right\rangle = \sigma^2 = \left(a^2+b^2\right).\end{aligned}$$

Notice that the variance is the addition of the variance of the particle and the meter.

b) By substituting the functions in the formula for $p(x|y)$ we get

$$\begin{aligned}p(x|y) &= \sqrt{\frac{(a^2+b^2)}{2\pi a^2b^2}}\exp\left[\frac{1}{2}\frac{(y-x_0)^2}{(a^2+b^2)}\right]\exp\left[-\frac{(x-x_0)^2}{2a^2}-\frac{(y-x)^2}{2b^2}\right],\\
&= \sqrt{\frac{(a^2+b^2)}{2\pi a^2b^2}}\exp\left[-\frac{(x-x_0)^2}{2a^2}-\frac{(y-x)^2}{2b^2}+\frac{1}{2}\frac{(y-x_0)^2}{(a^2+b^2)}\right].\end{aligned}$$

To calculate $\langle x\rangle_y$ we can just do the integral by brute force or we can use the method that Feynman would prefer by differentiation under the integral. Notice that

$$\begin{aligned}\frac{dp(y)}{dx_0} &= \int_{-\infty}^{\infty} dx\,|\psi(x)|^2\,|\varphi(y-x)|^2\left[\frac{(x-x_0)}{a^2}\right] \\ &= \int_{-\infty}^{\infty} dx\,|\psi(x)|^2\,|\varphi(y-x)|^2\left(\frac{x}{a^2}\right) - p(y)\frac{x_0}{a^2},\end{aligned}$$

also from the explicit form of $p(y)$ we get

$$\frac{dp(y)}{dx_0} = \frac{(y-x_0)}{(a^2+b^2)}p(y)$$

from this we get that

$$\begin{aligned}\langle x\rangle_y &= \frac{1}{p(y)}\int_{-\infty}^{\infty} dx\,|\psi(x)|^2\,|\varphi(y-x)|^2\,x \\ &= \frac{1}{p(y)}\left[a^2\frac{dp(y)}{dx_0} + p(y)x_0\right] \\ &= a^2\frac{(y-x_0)}{(a^2+b^2)} + x_0 \\ &= \frac{a^2y+b^2x_0}{(a^2+b^2)}.\end{aligned}$$

Then

$$\langle x\rangle_y = \frac{a^2y+b^2x_0}{(a^2+b^2)}.$$

For $\langle x^2\rangle_y$ we proceed as follow

$$\begin{aligned}\frac{d}{da}[ap(y)] &= \frac{1}{2\pi b}\int_{-\infty}^{\infty} dx\exp\left[-(x-x_0)^2/2a^2-(y-x)^2/2b^2\right]\left[\frac{(x-x_0)^2}{a^3}\right] \\ &= \frac{1}{2\pi ab}\int_{-\infty}^{\infty} dx\exp\left[-(x-x_0)^2/2a^2-(y-x)^2/2b^2\right]\left[\frac{x^2-2xx_0+x_0^2}{a^2}\right] \\ &= \frac{p(y)}{a^2}\langle x^2\rangle_y - 2\frac{x_0}{a^2}\langle x\rangle_y p(y) + p(y)\frac{x_0^2}{a^2}, \\ &= \frac{p(y)}{a^2}\left[\langle x^2\rangle_y - 2x_0\langle x\rangle_y + x_0^2\right].\end{aligned}$$

Also by using the explicit form of $p(y)$ we get

$$\begin{aligned}\frac{d}{da}[ap(y)] &= p(y) - \frac{a^2}{(a^2+b^2)}p(y) + p(y)\frac{a^2(y-x_0)^2}{(a^2+b^2)^2} \\ &= p(y)\left[\frac{b^2}{(a^2+b^2)} + \frac{a^2(y-x_0)^2}{(a^2+b^2)^2}\right], \\ &= p(y)\left[\frac{b^2}{(a^2+b^2)} + \frac{1}{a^2}\left(\langle x\rangle_y - x_0\right)^2\right],\end{aligned}$$

Combining both results we get

$$\left[\left\langle x^2\right\rangle_y - 2x_0\left\langle x\right\rangle_y + x_0^2\right] = \left[\frac{a^2b^2}{(a^2+b^2)} + \left(\left\langle x\right\rangle_y - x_0\right)^2\right].$$

$$\begin{aligned}\Rightarrow \quad \left\langle x^2\right\rangle_y &= \frac{a^2b^2}{(a^2+b^2)} + \left(\left\langle x\right\rangle_y - x_0\right)^2 + 2x_0\left\langle x\right\rangle_y - x_0^2 \\ = \quad \left\langle x^2\right\rangle_y &= \frac{a^2b^2}{(a^2+b^2)} + \left\langle x\right\rangle_y^2\end{aligned}$$

Finally,

$$\left[(\Delta x)^2\right]_y = \left\langle x^2\right\rangle_y - \left\langle x\right\rangle_y^2 = \frac{a^2b^2}{(a^2+b^2)}.$$

For a narrow meter $b^2 \ll a^2$ we have

$$\left\langle x\right\rangle_y \simeq y,$$

$$\left[(\Delta x)^2\right]_y \simeq b^2,$$

this make sense since for a narrow meter after the measurement we are going to know that the particle is around y with the same uncertainty or "resolution" that the meter has. This is, $p(x|y) \simeq |\varphi(x-y)|^2$ for a narrow meter as expected.

For a broad meter $b^2 \gg a^2$ we have

$$\left\langle x\right\rangle_y \simeq x_0 = \left\langle y\right\rangle,$$

$$\left[(\Delta x)^2\right]_y \simeq a^2 = (\Delta y)^2,$$

this make sense since a broad meter cannot give us better information than the one we already know about $\psi(x)$ from a). This is, $p(x|y) \simeq |\psi(x)|^2$ for a broad meter as expected.