Homework 6 Solutions

Ph 12b Winter 2010

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1. Geometric Phase

a. Simply plug-and-chug:

$$\hat{D}(\beta)\hat{D}(\alpha) = \exp\underbrace{\left(\beta\hat{a}^\dagger - \beta^*\hat{a}\right)}_{\hat{A}}\exp\underbrace{\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)}_{\hat{B}}$$

$$\hat{A}+\hat{B} = \beta\hat{a}^\dagger - \beta^*\hat{a} + \alpha\hat{a}^\dagger - \alpha^*\hat{a} = (\beta+\alpha)\hat{a}^\dagger - (\beta+\alpha)^*\hat{a} = \hat{D}(\beta+\alpha)$$

$$\frac{1}{2}[\hat{A},\hat{B}] = -\alpha^*\beta\hat{a}^\dagger\hat{a} - \alpha\beta^*\hat{a}\hat{a}^\dagger + \alpha\beta^*\hat{a}^\dagger\hat{a} + \alpha^*\beta * \hat{a}\hat{a}^\dagger = \alpha^*\beta - \alpha\beta^* \Rightarrow e^{i\phi(\beta,\alpha)} = e^{i\frac{1}{2i}\alpha^*\beta-\alpha\beta^*}$$

$$\Rightarrow \phi(\beta,\alpha) = \text{Im}[\alpha^*\beta]$$

b. Using the previous result as the hint suggests, the incremental change in the displacement can be represented by

$$\hat{D}(d\alpha)\hat{D}(\alpha) = e^{i\phi}\hat{D}(\alpha+d\alpha), \text{ where } \phi = \text{Im}[\alpha^* d\alpha] = \text{Im}[(\alpha - i\alpha)(\alpha + id\alpha)] = -\alpha_2 d\alpha_1 + \alpha_1 d\alpha_2$$

Since $\phi(P)$ is the total phase accumulated over path P, we get that

$$-\alpha_2 d\alpha_1 + \alpha_1 d\alpha_2 \Rightarrow \phi(P) = \int_P (-\alpha_2 d\alpha_1 + \alpha_1 d\alpha_2) \Rightarrow A_1 = -\alpha_2,\ A_2 = \alpha_1$$

c. Stoke's theorem (generalized Green's theorem) states that $\int_C \mathbf{A}\cdot d\vec{\alpha} = \int_S (\nabla\times\mathbf{A})d\vec{\alpha}$. From the result in part b), we see that $\phi(P) = \int_C \mathbf{A}\cdot d\vec{\alpha}$, where $\mathbf{A} = (-\alpha_2, \alpha_1)$, which implies $\nabla\times\mathbf{A} = 2$, or

$$\exp[i\phi(P)] = \exp\left[2i\int_S d\alpha_1 d\alpha_2\right],\ c = 2.$$

The sign of the integral depends how C encloses S.

d. We need to find the total displacement $\hat{D}(\alpha)$:

$$\hat{H}(t) = \hbar\Omega(-i)(\hat{a}e^{-i\nu t} - \hat{a}^\dagger e^{i\nu t}) \Rightarrow \hat{U}(t+dt, dt) = \exp(-dt\Omega(\hat{a}e^{-i\nu t} - \hat{a}^\dagger e^{i\nu t}))$$

$$\hat{D}(\alpha) = \int_0^t \hat{U}dt = \exp\left[\int_0^t \Omega(\hat{a}^\dagger e^{i\nu t} - \hat{a}e^{-i\nu t})dt\right] \quad \text{since } \hat{D}(\alpha) = \left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right),\ \alpha = \int_0^t \Omega e^{i\nu t}dt$$

$$\Rightarrow \alpha(t) = \frac{\Omega}{i\nu}\left(e^{i\nu t} - 1\right),\ \alpha\left(\frac{2\pi}{\nu}\right) = 0, \text{ so the cumulative displacement is } 0$$

$$\text{From part b), we know that Area} = \frac{1}{2}\int_P \alpha_1 d\alpha_2 - \alpha_2 d\alpha_1$$

$$\alpha_1 = \frac{\Omega}{\nu}\sin\nu t;\ \frac{d\alpha_1}{dt} = \Omega\cos\nu t \qquad \alpha_2 = \frac{\Omega}{\nu}(1-\cos\nu t);\ \frac{d\alpha_2}{dt} = \Omega\sin\nu t$$

$$\Rightarrow A = \frac{1}{2}\int_0^{2\pi/\nu} \frac{\Omega}{\nu}\sin\nu t \cdot \Omega \sin\nu t dt - \frac{\Omega}{\nu}(1-\cos\nu t)\cdot\Omega\cos\nu t dt = \pi\frac{\Omega^2}{\nu^2}$$

$$\Rightarrow \text{Geometric Phase: } e^{i2\pi\Omega^2/\nu^2}$$

e. First evaluate $\sigma_3 \otimes I + I \otimes \sigma_3$:

$$\sigma_3 \otimes I + I \otimes \sigma_3 = \begin{cases} 0 & |0\rangle \otimes |1\rangle,\ |1\rangle \otimes |0\rangle \text{ and the rest} \\ 2 & |0\rangle \otimes |0\rangle \\ -2 & |1\rangle \otimes |1\rangle \end{cases}$$

Applying the above to the two-qubit unitary operator $\hat{V}$ is similar to what we got above for d). The area for these 3 possible conditions is 0 and $4\pi\Omega^2/\nu^2$. Then the eigenvalues for $\hat{V}$ are $\left(e^{8\pi\Omega^2\nu^2}, 1, 1, e^{8\pi\Omega^2\nu^2}\right)$. If $\nu = 4\Omega$, the eigenvalues reduce to $(i, 1, 1, i)$ and give time $t = \pi/4\Omega$.

2. Squeezing an Oscillator

Note: all operators can commute with themselves (or powers thereof)!

a. This is simple enough to prove as a general equality:

$$(e^{\epsilon\hat{B}})^\dagger = \left(\sum_n^\infty \frac{(\epsilon\hat{B})^n}{n!}\right)^\dagger = \sum_n^\infty \frac{(\epsilon\hat{B}^\dagger)^n}{n!} = \sum_n^\infty \frac{(-\epsilon\hat{B})^n}{n!} = e^{-\epsilon\hat{B}} = (e^{\epsilon\hat{B}})^{-1}$$

and is thus unitary.

b. Find the Taylor expansion around $\epsilon = 0$:

$$\hat{G}(\epsilon) = \left(e^{\epsilon\hat{B}}\right)^\dagger \hat{A}\left(e^{\epsilon\hat{B}}\right) = \left(e^{-\epsilon\hat{B}}\right)\hat{A}\left(e^{\epsilon\hat{B}}\right) \Rightarrow \hat{G}(\epsilon) = \hat{G}(0) + \epsilon\hat{G}'(0) + \cdots$$
$$= \hat{A} - \epsilon\hat{B}e^{-\epsilon\hat{B}}\hat{A}e^{\epsilon\hat{B}} + \epsilon e^{-\epsilon\hat{B}}\hat{A}\hat{B}e^{\epsilon\hat{B}}\Big|_0 + \cdots = \hat{A} - \epsilon\hat{B}\hat{A} + \epsilon\hat{A}\hat{B} + \cdots = \hat{A} + \epsilon[\hat{A}, \hat{B}]$$

c. Use the result from b) to speed up the process. Set $\epsilon = r/2$ $(\epsilon/2)$, $\hat{A} = \hat{a}$ $(\hat{a}^\dagger)$ and $\hat{B} = \hat{a}^2 - (\hat{a}^\dagger)^2$:

$$\hat{S}(\epsilon)^\dagger\hat{a}\hat{S}(\epsilon) = \hat{a} + \frac{\epsilon}{2}\left[\hat{a}, \hat{a}^2 - (\hat{a}^\dagger)^2\right] = \hat{a} - \frac{\epsilon}{2}\left[\hat{a}, (\hat{a}^\dagger)^2\right] = \hat{a} + \frac{\epsilon}{2}\{\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}, \hat{a}^\dagger]\hat{a}^\dagger\} = \hat{a} - \epsilon a^\dagger$$
$$\hat{S}(\epsilon)^\dagger\hat{a}^\dagger\hat{S}(\epsilon) = \hat{a}^\dagger + \frac{\epsilon}{2}\left[\hat{a}^\dagger, \hat{a}^2 - (\hat{a}^\dagger)^2\right] = \hat{a}^\dagger - \frac{\epsilon}{2}\left[\hat{a}^\dagger, \hat{a}^2\right] = \hat{a}^\dagger + \frac{\epsilon}{2}\{\hat{a}[\hat{a}^\dagger, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a}\} = \hat{a}^\dagger - \epsilon a$$

d. Manipulate the operators such that above expansions can be used:

$$\hat{S}(\epsilon)^\dagger\hat{\xi}\hat{S}(\epsilon) = \hat{S}(\epsilon)^\dagger\frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)\hat{S}(\epsilon) = \frac{1}{\sqrt{2}}\{\hat{S}(\epsilon)^\dagger\hat{a}\hat{S}(\epsilon) + \hat{S}(\epsilon)^\dagger\hat{a}^\dagger\hat{S}(\epsilon)\} = \frac{1}{\sqrt{2}}\{(\hat{a} + \hat{a}^\dagger) - \epsilon(\hat{a} + \hat{a}^\dagger)\}$$
$$= (1 - \epsilon)\hat{\xi}$$
$$\hat{S}(\epsilon)^\dagger\hat{p}_\xi\hat{S}(\epsilon) = \hat{S}(\epsilon)^\dagger\frac{-i}{\sqrt{2}}(\hat{a} - \hat{a}^\dagger)\hat{S}(\epsilon) = \frac{-i}{\sqrt{2}}\{\hat{S}(\epsilon)^\dagger\hat{a}\hat{S}(\epsilon) - \hat{S}(\epsilon)^\dagger\hat{a}^\dagger\hat{S}(\epsilon)\} = \frac{-i}{\sqrt{2}}\{(\hat{a} - \hat{a}^\dagger) + \epsilon(\hat{a} - \hat{a}^\dagger)\}$$
$$= (1 + \epsilon)\hat{p}_\xi$$

e. Subsituting r for ϵ and using the limit definition of e,

$$\lim_{N\to\infty} = \left(1 - \frac{r}{N}\right)^N \hat{\xi} = e^{-r}\hat{\xi} \quad\text{and}\quad \lim_{N\to\infty} = \left(1 + \frac{r}{N}\right)^N \hat{p}_\xi = e^{r}\hat{p}_\xi$$

f. Subsitute:

$$\langle r|\hat{\xi}|r\rangle = \langle 0|\hat{S}(r)^\dagger\hat{\xi}\hat{S}(r)|0\rangle = \langle 0|e^{-r}\hat{\xi}|0\rangle = e^{-r}\langle 0|\hat{\xi}|0\rangle = 0 \text{ by definition}$$
$$\langle r|\hat{\xi}^2|r\rangle = e^{-2r}\langle 0|\hat{\xi}^2|0\rangle = \frac{e^{-2r}}{2}\langle 0|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|0\rangle = \frac{1}{2}e^{-2r}$$
$$\langle r|\hat{p}_\xi|r\rangle = e^{r}\langle 0|\hat{p}_\xi|0\rangle = 0$$
$$\langle r|\hat{p}_\xi^2|r\rangle = \frac{e^{2r}}{2}\langle 0|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|0\rangle = \frac{1}{2}e^{2r}$$

With this, we can calculate the variances:

$$(\Delta\hat{\xi})_r = \frac{1}{\sqrt{2}}e^{-r} \quad \text{and} \quad (\Delta\hat{p}_\xi)_r = \frac{1}{\sqrt{2}}e^{r}$$

which implies that the minimum uncertainty in this state has been "squeezed" to $\Delta\hat{\xi}\Delta\hat{p}_\xi = 1/2$.

3. Anharmonic Oscillator

a. Rewrite the energy expression to express the peturbative term:

$$E_n' - E_n = k\hbar\omega\langle n|\xi^4|n\rangle = \frac{1}{4}k\hbar\omega\langle n|(\hat{a}+\hat{a}^\dagger)^4|n\rangle$$

In the expansion for $(\hat{a}+\hat{a}^\dagger)^4$, only terms with two raising and lowering operators would contribute; all others would shift $|n\rangle$ into an orthogonal state. Thus we get

$$(\hat{a}+\hat{a}^\dagger)^4 = \hat{A} = \hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$$
$$\hat{a}|n\rangle = \sqrt{n+1}|n+1\rangle \qquad \hat{a}^\dagger|n\rangle = \sqrt{n}|n-1\rangle \qquad \hat{a}\hat{a}^\dagger|n\rangle = (n+1)|n\rangle \qquad \hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$$
$$\frac{1}{4}k\hbar\omega\langle n|\hat{A}|n\rangle = \frac{1}{4}k\hbar\omega\{(n+1)(n+2) + (n+1)^2 + n(n+1) + n(n+1) + n^2 + n(n-1)\}$$
$$\Rightarrow k\hbar\omega\langle n|\xi^4|n\rangle = \frac{1}{4}k\hbar\omega(6n^2+6n+3)$$

b. Evaluate the above expression:

$$E_{10}' = E_1' - E_0' = E_1 + \frac{15}{4}k\hbar\omega - E_0 - \frac{3}{4}k\hbar\omega = \hbar\omega + \frac{12}{4}k\hbar\omega$$
$$E_{21}' = E_2' - E_1' = E_2 + \frac{39}{4}k\hbar\omega - E_1 - \frac{15}{4}k\hbar\omega = \hbar\omega + \frac{24}{4}k\hbar\omega$$
$$\Delta(k) = \frac{E_{21}' - E_{10}'}{E_{10}'} = \frac{3k\hbar\omega}{\hbar\omega + 3k\hbar\omega}$$

This gives $3k$ after Taylor expanding with regards to k around 0 and choosing the first term.