PH12b 2010 Solutions HW#7

1.

a) Our state is

$$|\Psi(dt)\rangle = \sqrt{\Gamma dt}\hat{a}\,|\psi\rangle \otimes |1\rangle + \left(\hat{I} - \frac{1}{2}\Gamma dt \hat{a}^\dagger \hat{a}\right)|\psi\rangle \otimes |0\rangle$$

and

$$\langle \Psi(dt)| = \sqrt{\Gamma dt}\hat{a}^\dagger \langle\psi| \otimes \langle 1| + \left(\hat{I} - \frac{1}{2}\Gamma dt \hat{a}^\dagger \hat{a}\right)\langle\psi| \otimes \langle 0|$$

then

$$\begin{aligned}
\langle \Psi(dt)|\,\Psi(dt)\rangle &= \Gamma dt\,\langle\psi|\,\hat{a}^\dagger\hat{a}\,|\psi\rangle\,\langle 1|\,1\rangle + \langle\psi|\left(\hat{I} - \frac{1}{2}\Gamma dt \hat{a}^\dagger \hat{a}\right)^2|\psi\rangle\,\langle 0|\,0\rangle \\
&= \Gamma dt\,\langle\psi|\,\hat{a}^\dagger\hat{a}\,|\psi\rangle + \langle\psi|\,\psi\rangle - \Gamma dt\,\langle\psi|\,\hat{a}^\dagger\hat{a}\,|\psi\rangle + O(dt^2) \\
&= \langle\psi|\,\psi\rangle = 1.
\end{aligned}$$

b) The coherent state is

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,.$$

Then, working in linear order in dt we get

$$\begin{aligned}
\left(\hat{I} - \frac{1}{2}\Gamma dt \hat{a}^\dagger \hat{a}\right)|\alpha\rangle &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(\hat{I} - \frac{1}{2}\Gamma dt \hat{a}^\dagger \hat{a}\right)|n\rangle\,, \\
&= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(1 - \frac{1}{2}\Gamma dt n\right)|n\rangle\,, \\
&\approx e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(e^{-\Gamma dt/2}\right)^n|n\rangle\,, \\
&\approx e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\alpha e^{-\Gamma dt/2}\right]^n|n\rangle\,, \\
&\approx e^{-|\alpha|^2/2}\exp\left[\frac{1}{2}\left|\alpha e^{-\Gamma dt/2}\right|^2\right]\left|\alpha e^{-\Gamma dt/2}\right\rangle\,, \\
&\approx e^{-|\alpha|^2/2}\exp\left[\frac{1}{2}\left|\alpha\left(1 - \frac{1}{2}\Gamma dt\right)\right|^2\right]\left|\alpha e^{-\Gamma dt/2}\right\rangle\,, \\
&\approx \exp\left(-\frac{1}{2}\Gamma dt\,|\alpha|^2\right)\left|\alpha e^{-\Gamma dt/2}\right\rangle.
\end{aligned}$$

where we used $\hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle$ and

$$\left[e^{-\Gamma dt/2}\right] \approx \left(1 - \frac{1}{2}\Gamma dt\right) + O(dt^2).$$

c) We want to verify that

$$\Gamma dt\hat{a}\,|\alpha\rangle\,\langle\alpha|\,\hat{a}^\dagger \approx \Gamma dt\,|\alpha|^2\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\alpha e^{-\Gamma dt/2}\right|$$

to linear order in dt. First, because $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ the left-hand side gives

$$lhs = \Gamma dt\,|\alpha|^2\,|\alpha\rangle\,\langle\alpha|\,.$$

Now, for the right hand side we should expand $\left|\alpha e^{-\Gamma dt/2}\right\rangle$ to linear order in dt, this is

$$\begin{aligned}
\left|\alpha e^{-\Gamma dt/2}\right\rangle &= \exp\left[-\frac{1}{2}|\alpha|^2 e^{-\Gamma dt/2}\right]\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\alpha e^{-\Gamma dt/2}\right]^n|n\rangle \\
&\approx \exp\left[-\frac{1}{2}|\alpha|^2\left(1-\frac{1}{2}\Gamma dtn\right)\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(1-\frac{1}{2}\Gamma dtn\right)|n\rangle \\
&\approx e^{-|\alpha|^2/2}\left(1-\frac{1}{2}\Gamma dt\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(1-\frac{1}{2}\Gamma dtn\right)|n\rangle \approx |\alpha\rangle\left(1+O(\Gamma dt)\right).
\end{aligned}$$

Then

$$rhs = \Gamma dt\,|\alpha|^2\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\alpha e^{-\Gamma dt/2}\right| \approx \Gamma dt\,|\alpha|^2|\alpha\rangle\langle\alpha| = lhs.$$

Using this, the result in a) and Eq.(1) the evolution of $|\alpha\rangle\langle\alpha|$ is given by

$$\begin{aligned}
|\alpha\rangle\langle\alpha| &\to \Gamma dt\hat{a}|\alpha\rangle\langle\alpha|\hat{a}^\dagger + \left(\hat{I}-\frac{1}{2}\Gamma dt\hat{a}^\dagger\hat{a}\right)|\alpha\rangle\langle\alpha|\left(\hat{I}-\frac{1}{2}\Gamma dt\hat{a}^\dagger\hat{a}\right), \\
&\approx \Gamma dt\,|\alpha|^2\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\alpha e^{-\Gamma dt/2}\right| + \exp\left(-\frac{1}{2}\Gamma dt\,|\alpha|^2\right)^2\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\alpha e^{-\Gamma dt/2}\right|, \\
&\approx \Gamma dt\,|\alpha|^2\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\alpha e^{-\Gamma dt/2}\right| + (1-\Gamma dt\,|\alpha|^2)\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\alpha e^{-\Gamma dt/2}\right|, \\
&= \left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\alpha e^{-\Gamma dt/2}\right|.
\end{aligned}$$

then

$$|\alpha\rangle\langle\alpha| \to \left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\alpha e^{-\Gamma dt/2}\right|$$

to linear order in dt.

To evolve the state a finite time $t = ndt$ ($n >> 1$) we just have to evolve the state n times by applying the last formula n times. It is easy to see that this gives

$$|\alpha\rangle\langle\alpha| \to \left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\alpha e^{-\Gamma dt/2}\right| \to \left|\alpha e^{-\Gamma dt}\right\rangle\left\langle\alpha e^{-\Gamma dt}\right| \to \ldots \to \left|\alpha e^{-\Gamma(ndt)/2}\right\rangle\left\langle\alpha e^{-\Gamma(ndt)/2}\right| = \left|\alpha e^{-\Gamma t/2}\right\rangle\left\langle\alpha e^{-\Gamma t/2}\right|,$$

then the initial coherent state evolve as

$$|\alpha\rangle \to \left|\alpha e^{-\Gamma t/2}\right\rangle.$$

d)

$$\begin{aligned}
\langle\beta|\,\alpha\rangle &= \left[e^{-|\beta|^2/2}\sum_{m=0}^{\infty}\frac{\beta^{*m}}{\sqrt{m!}}\langle m|\right]\left[e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\right], \\
&= e^{-|\alpha|^2/2}e^{-|\beta|^2/2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\beta^{*m}}{\sqrt{m!}}\frac{\alpha^n}{\sqrt{n!}}\langle m|\,|n\rangle, \\
&= e^{-|\alpha|^2/2}e^{-|\beta|^2/2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\beta^{*m}}{\sqrt{m!}}\frac{\alpha^n}{\sqrt{n!}}\delta_{mn}, \\
&= e^{-|\alpha|^2/2}e^{-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{(\beta^*\alpha)^n}{n!}, \\
&= e^{-|\alpha|^2/2}e^{-|\beta|^2/2}e^{\beta^*\alpha}, \\
&= \exp\left[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2-\beta^*\alpha-\alpha^*\beta\right)\right]\exp\left[\frac{1}{2}\left(\beta^*\alpha-\alpha^*\beta\right)\right], \\
&= \exp\left[-\frac{1}{2}|\alpha-\beta|^2\right]\exp\left[\frac{1}{2}\left(\beta^*\alpha-\alpha^*\beta\right)\right], \\
&= \exp\left[-\frac{1}{2}|\alpha-\beta|^2\right]\exp\left[i\,\mathrm{Im}\left(\alpha\beta^*\right)\right].
\end{aligned}$$

Then
$$\langle\beta|\ \alpha\rangle = e^{-|\alpha-\beta|^2/2}e^{i\,\mathrm{Im}(\alpha\beta^*)},$$
notice that the last factor is just a phase factor.

Now, we want to normalize
$$|\psi\rangle = N_{\alpha,\beta}\left(|\alpha\rangle + |\beta\rangle\right),$$
then
$$\begin{aligned}
1 &= \langle\psi|\ \psi\rangle = N^2_{\alpha,\beta}\left(\langle\alpha|\ \alpha\rangle + \langle\beta|\ \beta\rangle + \langle\alpha|\ \beta\rangle + \langle\beta|\ \alpha\rangle\right),\\
&= N^2_{\alpha,\beta}\left(2 + e^{-|\alpha-\beta|^2/2}\left[e^{i\,\mathrm{Im}(\alpha^*\beta)} + e^{i\,\mathrm{Im}(\alpha\beta^*)}\right]\right),\\
&= N^2_{\alpha,\beta}\left(2 + e^{-|\alpha-\beta|^2/2}2\cos\mathrm{Im}\left(\alpha\beta^*\right)\right),\\
&\Rightarrow\ N_{\alpha,\beta} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+e^{-|\alpha-\beta|^2/2}\cos\mathrm{Im}\left(\alpha\beta^*\right)}}.
\end{aligned}$$

e) We want to compute
$$|\alpha\rangle\langle\beta| \to \Gamma dt\hat{a}\,|\alpha\rangle\langle\beta|\,\hat{a}^\dagger + \left(\hat{I} - \frac{1}{2}\Gamma dt\hat{a}^\dagger\hat{a}\right)|\alpha\rangle\langle\beta|\left(\hat{I} - \frac{1}{2}\Gamma dt\hat{a}^\dagger\hat{a}\right),$$
using the results in c) and b) we can write it like
$$\begin{aligned}
|\alpha\rangle\langle\beta| &\to \Gamma dt\left(\alpha\beta^*\right)|\alpha\rangle\langle\beta| + e^{-\Gamma dt|\alpha|^2/2}e^{-\Gamma dt|\alpha|^2/2}\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\beta e^{-\Gamma dt/2}\right|,\\
&\approx \Gamma dt\left(\alpha\beta^*\right)\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\beta e^{-\Gamma dt/2}\right| + e^{-\Gamma dt|\alpha|^2/2}e^{-\Gamma dt|\alpha|^2/2}\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\beta e^{-\Gamma dt/2}\right|,\\
&\approx \Gamma dt\left(\alpha\beta^*\right)\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\beta e^{-\Gamma dt/2}\right| + \left[1 - \Gamma dt\left|\alpha\right|^2/2 - \Gamma dt\left|\beta\right|^2/2 + O(\Gamma^2dt^2)\right]\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\beta e^{-\Gamma dt/2}\right|,\\
&\approx \left[1 - \frac{\Gamma dt}{2}\left(\left|\alpha\right|^2 + \left|\beta\right|^2 - 2\alpha\beta^*\right)\right]\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\beta e^{-\Gamma dt/2}\right|,\\
&\approx \left[1 - \frac{\Gamma dt}{2}\left(\left|\alpha-\beta\right|^2 - \alpha\beta^* + \alpha^*\beta\right)\right]\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\beta e^{-\Gamma dt/2}\right|,\\
&\approx e^{i\Gamma dt\,\mathrm{Im}(\alpha\beta^*)}e^{-\Gamma dt|\alpha-\beta|^2/2}\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\beta e^{-\Gamma dt/2}\right|,
\end{aligned}$$
then
$$|\alpha\rangle\langle\beta| \to \left[e^{i\Gamma dt\,\mathrm{Im}(\alpha\beta^*)}e^{-\Gamma dt|\alpha-\beta|^2/2}\right]\left|\alpha e^{-\Gamma dt/2}\right\rangle\left\langle\beta e^{-\Gamma dt/2}\right|,$$

Using the same procedure as in the last part of c) it is easy to see that for finite time t the state will evolve to
$$|\alpha\rangle\langle\beta| \to \left[e^{i\phi(t)}e^{-\sigma(t)}\right]\left|\alpha e^{-\Gamma t/2}\right\rangle\left\langle\beta e^{-\Gamma t/2}\right|,$$
where $\sigma(t)$, $\phi(t)$, are some real functions of t that we will have to determined. So far we know that $\phi(0) = 0$, and $\sigma(0) = 0$. Now, evolve the state at t one more time a small dt we get
$$\left[e^{i\phi(t)}e^{-\sigma(t)}\right]\left|\alpha e^{-\Gamma t/2}\right\rangle\left\langle\beta e^{-\Gamma t/2}\right| \to \exp\left[i\left(\phi(t) + \Gamma dte^{-\Gamma t}\,\mathrm{Im}(\alpha\beta^*)\right)\right]\exp\left[-\left(\sigma(t) + \Gamma dte^{-\Gamma t}\left|\alpha-\beta\right|^2/2\right)\right]\left|\alpha e^{-\Gamma(t+dt)/2}\right\rangle\left\langle\beta\right.$$
then we can say that
$$\begin{aligned}
\phi(t+dt) &= \phi(t) + \Gamma dte^{-\Gamma t}\,\mathrm{Im}(\alpha\beta^*),\\
\sigma(t+dt) &= \sigma(t) + \Gamma dte^{-\Gamma t}\left|\alpha-\beta\right|^2/2,
\end{aligned}$$
or
$$\frac{d\phi(t)}{dt} = \Gamma e^{-\Gamma t}\,\mathrm{Im}(\alpha\beta^*),\qquad \frac{d\sigma(t)}{dt} = \Gamma e^{-\Gamma t}\left|\alpha-\beta\right|^2/2,$$

solving this equations we get

$$\begin{aligned}\phi(t) &= \mathrm{Im}(\alpha\beta^*)\left(1-e^{-\Gamma t}\right), \\ \sigma(t) &= \frac{|\alpha-\beta|^2}{2}\left(1-e^{-\Gamma t}\right)\end{aligned}$$

f) We know that

$$\alpha = \sqrt{\frac{m\omega}{2\hbar}}x, \quad \beta = -\sqrt{\frac{m\omega}{2\hbar}}x,$$

then

$$(\alpha-\beta) = \sqrt{\frac{2m\omega}{\hbar}}x \Rightarrow \frac{|\alpha-\beta|^2}{2} = \frac{m\omega}{\hbar}x^2.$$

From e) we know that

$$\Gamma_{decohere} = \frac{|\alpha-\beta|^2}{2}\Gamma_{damp} = \frac{m\omega^2}{\hbar Q}x^2.$$

Now using, $\omega = 1\left[s^{-1}\right]$, $m = 0.001\,[kg]$, $x = \pm 0.01\,[m]$, $\hbar = 10^{-34}\left[m^2kg/s\right]$, we get

$$\Gamma_{decohere} \sim 10^{18}s^{-1}.$$

This is lager than Γ_{damp} by a factor of 10^{27}.

2.

a)

$$\begin{aligned}\varphi(x) &= Ae^{ikx} + Be^{-ikx}, \qquad x<0, \\ \varphi(x) &= Ce^{ikx} + De^{-ikx}, \qquad x<0,\end{aligned}$$

The first boundary condition imply

$$A + B = C + D,$$

and the second one

$$\begin{aligned}ik(A-B) &= ik(C-D) + 2\Delta(A+B), \\ A-B &= C-D-2i\frac{\Delta}{k}(A+B), \\ &\Rightarrow\; C-D = (A-B) + 2i\frac{\Delta}{k}(A+B),\end{aligned}$$

where $\alpha = \Delta/k$, solving these equations we get

$$\begin{aligned}C &= (1+i\alpha)A + i\alpha B, \\ D &= (1-i\alpha)B - i\alpha A,\end{aligned}$$

then

$$\begin{aligned}\begin{pmatrix} C \\ D \end{pmatrix} &= \begin{pmatrix} 1+i\alpha & i\alpha \\ -i\alpha & 1-i\alpha \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}, \\ M(\alpha) &= \begin{pmatrix} 1+i\alpha & i\alpha \\ -i\alpha & 1-i\alpha \end{pmatrix}.\end{aligned}$$

b) In general

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

since $\det M = 1$ then

$$M^{-1}(\alpha) = \begin{pmatrix} 1-i\alpha & -i\alpha \\ i\alpha & 1+i\alpha \end{pmatrix}.$$

c) We have that

$$\begin{aligned} \begin{pmatrix} A \\ B \end{pmatrix} &= \begin{pmatrix} 1-i\alpha & -i\alpha \\ i\alpha & 1+i\alpha \end{pmatrix}\begin{pmatrix} C \\ 0 \end{pmatrix} = \begin{pmatrix} 1-i\alpha \\ i\alpha \end{pmatrix} C, \\ &\Rightarrow \frac{C}{A} = \frac{1}{1-i\alpha}, \qquad \frac{B}{A} = \frac{B}{C}\frac{C}{A} = \frac{i\alpha}{1-i\alpha}. \end{aligned}$$

Then,

$$\begin{aligned} T(\alpha) &= \left|\frac{C}{A}\right|^2 = \frac{1}{1+\alpha^2}, \\ R(\alpha) &= \left|\frac{B}{A}\right|^2 = \frac{\alpha^2}{1+\alpha^2}, \end{aligned}$$

and

$$R + T = \frac{1}{1+\alpha^2} + \frac{\alpha^2}{1+\alpha^2} = 1,$$

as expected.

d) We want

$$\frac{C}{A} = \frac{1}{1-i\alpha},$$

to diverge, then $1 - i\alpha = 0 \Rightarrow \alpha = -i = \Delta/k = -i\Delta/\kappa$, where $k = i\kappa$ then

$$\kappa = \Delta.$$

In this case the normalized bound state solution is given by

$$\begin{aligned} \varphi(x) &= \sqrt{\kappa}e^{\kappa x}, \qquad x < 0, \\ \varphi(x) &= \sqrt{\kappa}e^{-\kappa x}, \qquad x < 0, \end{aligned}$$

3.

a) This is the same procedure that we already did in 2a), you can solve the problem by brute force doing the same calculations or you can to used the previews results as follow:

First, the general solution is

$$\begin{aligned} \varphi(x) &= \bar{A}e^{ikx} + \bar{B}e^{-ikx}, \qquad x < -a, \\ \varphi(x) &= \bar{C}e^{ikx} + \bar{D}e^{-ikx}, \qquad -a < x < a, \\ \varphi(x) &= \bar{E}e^{ikx} + \bar{F}e^{-ikx}, \qquad x > a. \end{aligned}$$

notice that we used an overbar in the coefficients. To derive M notice that the only difference with the problem in 2a) is that the $\delta-$function potential is at $x = -a$, then we only need to shift $x \to x + a$ in 2a) and relate the coefficients to solve our problem. When we shift $x \to x + a$ we get the following relations: $\bar{A} = Ae^{ika}$, $\bar{B} = Be^{-ika}$, $\bar{C} = Ce^{ika}$, $\bar{D} = De^{-ika}$. Then, using M from 2a) and making the substitutions we get

$$\begin{aligned} \begin{pmatrix} \bar{C}e^{-ika} \\ \bar{D}e^{ika} \end{pmatrix} &= \begin{pmatrix} 1+i\alpha & i\alpha \\ -i\alpha & 1-i\alpha \end{pmatrix}\begin{pmatrix} \bar{A}e^{-ika} \\ \bar{B}e^{ika} \end{pmatrix}, \\ &\Rightarrow \begin{pmatrix} e^{-ika} & 0 \\ 0 & e^{ika} \end{pmatrix}\begin{pmatrix} \bar{C} \\ \bar{D} \end{pmatrix} = \begin{pmatrix} 1+i\alpha & i\alpha \\ -i\alpha & 1-i\alpha \end{pmatrix}\begin{pmatrix} e^{-ika} & 0 \\ 0 & e^{ika} \end{pmatrix}\begin{pmatrix} \bar{A} \\ \bar{B} \end{pmatrix}, \\ &\Rightarrow \begin{pmatrix} \bar{C} \\ \bar{D} \end{pmatrix} = \begin{pmatrix} e^{ika} & 0 \\ 0 & e^{-ika} \end{pmatrix}\begin{pmatrix} 1+i\alpha & i\alpha \\ -i\alpha & 1-i\alpha \end{pmatrix}\begin{pmatrix} e^{-ika} & 0 \\ 0 & e^{ika} \end{pmatrix}\begin{pmatrix} \bar{A} \\ \bar{B} \end{pmatrix}, \\ &\Rightarrow \begin{pmatrix} \bar{C} \\ \bar{D} \end{pmatrix} = \begin{pmatrix} 1+i\alpha & i\alpha e^{i2ka} \\ -i\alpha e^{-i2ka} & 1-i\alpha \end{pmatrix}\begin{pmatrix} \bar{A} \\ \bar{B} \end{pmatrix}, \end{aligned}$$

then

$$M(\alpha,a)=\begin{pmatrix}1+i\alpha & i\alpha e^{i2ka}\\ -i\alpha e^{-i2ka} & 1-i\alpha\end{pmatrix}.$$

To get $N(\alpha,a)$ just notice that we have to shift now $x\to x-a$, instead of $x\to x+a$, and ($\bar{A}\to\bar{C}$, $\bar{B}\to\bar{D}$, $\bar{D}\to\bar{F}$, $\bar{C}\to\bar{E}$) then $N(\alpha,a)=M(\alpha,-a)$, this is

$$\begin{aligned}\begin{pmatrix}\bar{E}\\ \bar{F}\end{pmatrix} &= N(\alpha,a)\begin{pmatrix}\bar{C}\\ \bar{D}\end{pmatrix},\\ N(\alpha,a) &= \begin{pmatrix}1+i\alpha & i\alpha e^{-i2ka}\\ -i\alpha e^{i2ka} & 1-i\alpha\end{pmatrix}.\end{aligned}$$

The inverse matrices are given by (since $\det M=1=\det N$)

$$\begin{aligned}M^{-1}(\alpha,a) &= \begin{pmatrix}1-i\alpha & -i\alpha e^{i2ka}\\ i\alpha e^{-i2ka} & 1+i\alpha\end{pmatrix},\\ N^{-1}(\alpha,a) &= \begin{pmatrix}1-i\alpha & -i\alpha e^{-i2ka}\\ i\alpha e^{i2ka} & 1+i\alpha\end{pmatrix}.\end{aligned}$$

b) We have that

$$\begin{aligned}\begin{pmatrix}\bar{A}\\ \bar{B}\end{pmatrix} &= M^{-1}\begin{pmatrix}\bar{C}\\ \bar{D}\end{pmatrix},\\ &= M^{-1}N^{-1}\begin{pmatrix}\bar{E}\\ 0\end{pmatrix},\\ &= \begin{pmatrix}1-i\alpha & -i\alpha e^{i2ka}\\ i\alpha e^{-i2ka} & 1+i\alpha\end{pmatrix}\begin{pmatrix}1-i\alpha & -i\alpha e^{-i2ka}\\ i\alpha e^{i2ka} & 1+i\alpha\end{pmatrix}\begin{pmatrix}\bar{E}\\ \bar{F}\end{pmatrix},\\ &= \begin{pmatrix}1+\alpha\left(-2i+(-1+e^{4ika})\alpha\right) & 2i\alpha\left(-\cos(2ka)+\alpha\sin(2ka)\right)\\ 2i\alpha\left(\cos(2ka)-\alpha\sin(2ka)\right) & 1+\alpha\left(2i+(-1+e^{-4ika})\alpha\right)\end{pmatrix}\begin{pmatrix}\bar{E}\\ 0\end{pmatrix},\\ &= \begin{pmatrix}1+\alpha\left(-2i+(-1+e^{4ika})\alpha\right)\\ 2i\alpha\left(\cos(2ka)-\alpha\sin(2ka)\right)\end{pmatrix}\bar{E}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\bar{B}}{\bar{E}} &= 2i\alpha\left(\cos(2ka)-\alpha\sin(2ka)\right),\\ &\Rightarrow \left|\frac{B}{E}\right|^2=\frac{R}{T}=4\alpha^2\left(\cos(2ka)-\alpha\sin(2ka)\right)^2.\end{aligned}$$

Now, because $R+T=1$,

$$\begin{aligned}\frac{R}{T} &= \frac{1-T}{T}=\frac{1}{T}-1,\\ &\Rightarrow \frac{1}{T}=\frac{R}{T}+1,\\ &= 4\alpha^2\left(\cos(2ka)-\alpha\sin(2ka)\right)^2+1.\end{aligned}$$

c) First, let's simplify the result for $1/T$.

$$\begin{aligned}\cos(2ak)-\alpha\sin(2ak) &= \sqrt{1+\alpha^2}\left[\frac{1}{\sqrt{1+\alpha^2}}\cos 2ka-\frac{\alpha}{\sqrt{1+\alpha^2}}\sin 2ka\right],\\ &= \sqrt{1+\alpha^2}\left[\cos\theta\cos 2ka-\sin\theta\sin 2ka\right],\\ &= \sqrt{1+\alpha^2}\cos(2ka+\theta),\end{aligned}$$

where $\tan\theta = \alpha$. Then,

$$\frac{1}{T} = 1 + 4\alpha^2 \left(1+\alpha^2\right) \cos^2\left(2ka+\theta\right).$$

From this simplify expression we get

$$\begin{aligned}
T_{\max} \quad &\Rightarrow \quad \frac{1}{T}\min \Rightarrow \cos^2\left(2ka+\theta\right) = 0, \\
&\Rightarrow \quad 2ka+\theta = (2n+1)\frac{\pi}{2}, \qquad n = 0,1,2,3\ldots, \\
&\Rightarrow \quad a_{\max} = \frac{1}{2k}\left[(2n+1)\frac{\pi}{2} - \arctan\alpha\right],
\end{aligned}$$

and

$$\begin{aligned}
T_{\min} \quad &\Rightarrow \quad \frac{1}{T}\max \Rightarrow \cos^2\left(2ka+\theta\right) = \pm 1, \\
&\Rightarrow \quad 2ka+\theta = n\pi, \qquad n = 0,1,2,3\ldots, \\
&\Rightarrow \quad a_{\min} = \frac{1}{2k}\left[n\pi - \arctan\alpha\right].
\end{aligned}$$