Homework 8 Solutions

Ph 12b Winter 2010

March 13, 2010

1. A Barrier in a Well

a. For an even energy eigenstate ϕ, $\phi(x) = \phi(-x)$ and $\phi'(x) = -\phi'(-x)$. functions, From the previous problem set, we know

$$\lim_{\epsilon \to 0} \phi(x-\epsilon) - \phi(x+\epsilon) \Rightarrow \phi(0^-) - \phi(0^+) = 0 \Rightarrow \phi(0^-) = \phi(0^+) \Rightarrow \phi(0^-),\ \phi(0^+) \to \phi(0)$$

The derivative matching condition gives us

$$\phi'(0^+) - \phi'(0^-) = 2\Delta\phi(0) \Rightarrow 2\phi'(0^+) = 2\Delta\phi(0) \Rightarrow \frac{\phi'(0^+)}{\phi(0^+)} = \Delta$$

Similarly,

$$\frac{\phi'(0^-)}{\phi(0^-)} = -\Delta.$$

b. Assume the same functional form of ϕ as last time and use the boundary conditions $\phi(a) = \phi(-a) = 0$ to get

$$\phi(x) = \phi(-x) \Rightarrow Ce^{ikx} + De^{-ikx} = Ae^{-ikx} + Be^{ikx} \Rightarrow A = D,\ B = C$$
$$\Rightarrow \phi'(0^+) - \phi'(0^-) = ikC - ikD - ikA + ikB = 2ik(B - A) = 2\Delta(A + B) = 2\Delta\phi(0)$$
$$\Rightarrow B = \frac{ik + \Delta}{ik - \Delta} A$$
$$\phi(-a) = Ae^{-ika} + Be^{ika} = Ae^{-ika} + \frac{ik+\Delta}{ik-\Delta} Ae^{ika} = 0$$
$$ik(e^{-ika} + e^{ika}) - \Delta(e^{-ika} - e^{ika}) = 0 \Rightarrow 2ik\cos(ka) + 2i\Delta\sin(ka) \Rightarrow \Delta = -k\cot(ka)$$
$$\Rightarrow \Delta a = -ka\cot(ka)$$

c. For n odd, the delta barrier does not affect the wavefunction since since $\phi(0) = 0$. As $\Delta a \to \infty$, $\sin(ka) \to 0$, meaning that $k = o\pi/a$. Let o be the number of nodes; then $o = n + 1/2$ and

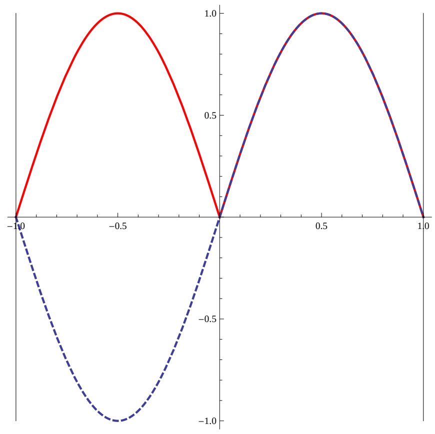
$$E_n = \frac{\hbar^2\pi^2}{2ma^2}\left(\frac{n+1}{2}\right)^2.$$

For n even, the barrier has an effect at $x = 0$ which requires the creation of an additional node, requiring that $o = n + 2/2$ and

$$E_n = \frac{\hbar^2\pi^2}{2ma^2}\left(\frac{n+2}{2}\right)^2$$

resulting in degeneracy.

d. The ground state must be even and is shown in red and the first excited state is in blue.



2. Reflectionless Potential

a.

$$\hat{H}\psi(x) = E\psi(x) \;\Rightarrow\; \left[\frac{\hat{p}}{2m} + V(x)\right]\psi(k_0 x) = E\psi(k_0 x)$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{m}k_0^2\,\mathrm{sech}^2(k_0 x)\right]\phi(k_0 x) = E\phi(k_0 x) \qquad z = k_0 x \;\Rightarrow\; dz = k_0 dx$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}k_0^2 - \frac{\hbar^2}{m}k_0^2\,\mathrm{sech}^2(z)\right]\phi(z) = E\phi(z) \;\Rightarrow\; \left[-\frac{d^2}{dz^2} - 2\,\mathrm{sech}^2(z)\right]\phi(z) = \underbrace{\frac{2mE}{\hbar^2 k_0^2}}_{\bar{k}^2}\phi(z)$$

b.

$$\left[-\frac{d^2}{dz^2} - 2\,\mathrm{sech}^2(z)\right](i\bar{k} - \tanh z)e^{i\bar{k}z} = -\frac{d}{dz}\left[-\mathrm{sech}^2 z\, e^{i\bar{k}z} + i\bar{k}\phi(z)e^{i\bar{k}z}\right] - 2\,\mathrm{sech}^2 z\,\phi(z)$$

$$\Rightarrow\; (\bar{k}^2 + 2\,\mathrm{sech}^2 z)\phi(z) - 2\,\mathrm{sech}^2 z\,\phi(z) = \bar{k}^2\phi(z)$$

c. We only have to show the limiting behavior of $i\bar{k} - \tanh z$:

$$\lim_{z\to\infty} i\bar{k} - \tanh z = i\bar{k} - 1 = C \qquad\qquad \lim_{z\to-\infty} i\bar{k} - \tanh z = i\bar{k} + 1 = A$$

d.

$$\frac{C}{A} = \frac{i\bar{k} - 1}{i\bar{k} + 1} = \frac{(i\bar{k} - 1)(i\bar{k} - 1)}{(i\bar{k} + 1)(i\bar{k} - 1)} = \frac{\bar{k}^2 + 2i\bar{k} - 1}{\bar{k}^2 + 1}$$

$$T = \left|\frac{C}{A}\right|^2 = \frac{1}{(\bar{k}^2 + 1)^2}(\bar{k}^2 + 2i\bar{k} - 1)(\bar{k}^2 - 2i\bar{k} - 1) = \frac{\bar{k}^4 + 2\bar{k}^2 + 1}{(\bar{k}^2 + 1)^2} = 1$$

$$R = 1 - T = 0$$

e. Solve:

$$\frac{A}{C} = \frac{i\bar{k} + 1}{i\bar{k} - 1} = \frac{i(i\bar{\kappa}) + 1}{i(i\bar{\kappa}) - 1} = \frac{-\bar{\kappa} + 1}{-\bar{\kappa} - 1} = 0 \;\Rightarrow\; \bar{\kappa} = 1 \;\Rightarrow\; \bar{k} = i$$

f. $\bar{k}^2 = -\bar{\kappa}^2 = -1$

$$\left[-\frac{d^2}{dz^2} - 2\,\text{sech}^2(z)\right]\phi(z) = \frac{d}{dz}\,\text{sech}\,z\tanh z - 2\,\text{sech}^3 z = \text{sech}^3 z - \text{sech}\,z\tanh^2 z - 2\,\text{sech}^3 z$$
$$\Rightarrow\ -\,\text{sech}\,z[\,\text{sech}^2 z + \tanh^2 z] = -\,\text{sech}\,z = \bar{k}^2\phi(z)$$
$$\bar{k}^2 = \frac{2mE}{\hbar^2 k_0^2}\ \Rightarrow\ E = -\frac{\hbar^2 k_0^2}{2m}$$

3. Bound States in a Linear Potential

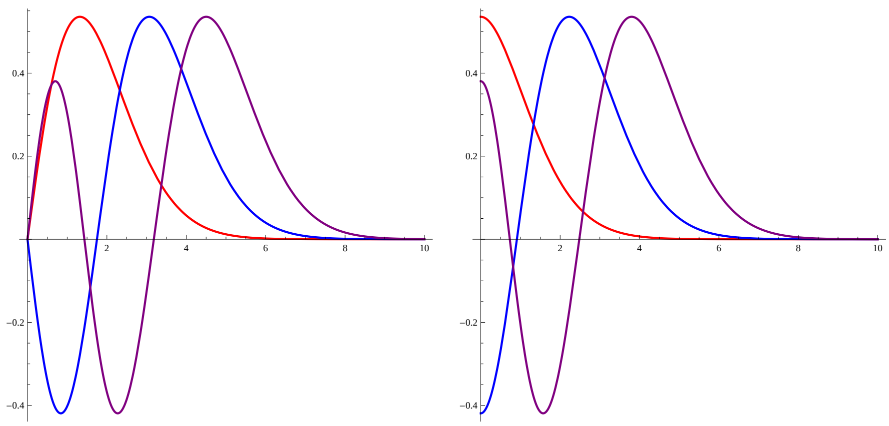
a.

$$\hat{H}\psi(x) = E\psi(x)\ \Rightarrow\ \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - F|x|\right]\phi(x) = E\phi(x) \qquad \text{for } x \geq 0 \text{ and } y = \left(\frac{\hbar^2}{2mF}\right)^{-1/3} x$$
$$\Rightarrow\ dy = \left(\frac{\hbar^2}{2mF}\right)^{-1/3} dx\ \Rightarrow\ \left[-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\left(\frac{2mF}{\hbar^2}\right)^{2/3} - F\left(\frac{\hbar^2}{2mF}\right)^{1/3} y\right]\phi(y) = E\phi(y)$$
$$\Rightarrow\ \left(\frac{\hbar^2 F^2}{2m}\right)^{1/3}\left[-\frac{d^2}{dy^2} + y\right]\phi(y) = E\phi(y)\ \Rightarrow\ \left[-\frac{d^2}{dy^2} + y\right]\phi(y) = \bar{E}\phi(y)$$

b. For an even solution $\phi'(0) = 0$ and for an odd solution $\phi(0) = 0$. Then we get

$$\phi(x) \sim \text{Ai}\,(\alpha x + \bar{E})\ \Rightarrow\ \phi(0) = \begin{cases} \text{Ai}'\,(\bar{E}) = 0 & \text{even solutions} \\ \text{Ai}\,(\bar{E}) = 0 & \text{odd solutions} \end{cases}$$

We can see how the Airy solutions look like when shifted by the zeros: When n is odd, there are



(a) Odd: Red = 1, Blue = 3, Purple = 5 (b) Even: Red = 0, Blue = 2, Purple = 4

$o = n - 1/2$ zeros on the right side excluding the one at the origin (see above plots). Then by symmetry, the whole wavefunction has $2o + 1 = n$ nodes. When n is even, there are $o = n/2$ zeros on the right side. By symmetry, the whole wavefunction has $2o = n$ nodes; thus the solutions have n nodes.

c.

$$\int_{x_1}^{x_2} k(x)\,dx = \pi\left(n + \frac{1}{2}\right) \qquad \text{classical turning points: } x_i = \pm\frac{E_n}{F}$$

$$k(x)^2 = \frac{2m}{\hbar^2}(E_n - V(x)) \;\Rightarrow\; k(x) = \sqrt{\frac{2m}{\hbar^2}(E_n - Fx)}$$

$$\Rightarrow\; 2\sqrt{\frac{2m}{\hbar^2}} \int_0^{E_n/F} \sqrt{E_n - Fx}\,dx = 2\sqrt{\frac{2m}{\hbar^2}} \left(-\frac{2}{3F}(E_n - Fx)^{3/2} \right)\Bigg|_0^{E_n/F} = \frac{4}{3F}\sqrt{\frac{2m}{\hbar^2}} E_n^{3/2} = \pi \left(n + \frac{1}{2} \right)$$

$$\Rightarrow\; E_n = \left(\frac{\hbar^2 F^2}{2m} \frac{9}{16} \pi^2 \left(n + \frac{1}{2} \right)^2 \right)^{1/3} = \bar{E}_n = \left(\frac{3}{4}\pi \left(n + \frac{1}{2} \right) \right)^{2/3}$$

We then get

n	WKB Prediction	Airy Exact	% difference
0	1.115	1.0188	+9.4%
1	2.320	2.3381	-0.8%
2	3.261	3.2482	+0.4%
3	4.082	4.0879	-0.14%
4	4.827	4.8201	+0.14%
5	5.517	5.5206	-0.06%

At $n = 19$, the 10^{th} of $\mathrm{Ai}\,(z)$ is 12.8288. The predicted value of $\bar{E}_n = 12.8281$, which is lower by $5{\times}10^{-5}$.