Ph 12b Quantum Physics
26 January 2010
Decoherence

Note Title 1/25/2010

Part of our motivation for developing the density matrix formalism is to prepare the way for a discussion of *decoherence*. Decoherence provides the explanation for why, although the fundamental laws of physics are quantum mechanical, macroscopic systems behave classically (i.e., they do not exhibit quantum interference). The underlying quantum behaviour is hidden when a system is imperfectly isolated from its environment, and so becomes correlated with the environment. While microscopic systems like individual electrons, photons, or atoms can be well enough isolated that quantum effects can be detected, for macroscopic systems the required degree of isolation is rarely achievable, and the laws of classical mechanics therefore apply to high accuracy.

For example, we have noted that double slit interference experiments can be performed with electrons or with photons, and even with (C_60) buckyballs, but not (yet) with bacteria or dust particles. Why not? Suppose we prepare a quantum state of a dust particle in a superposition of two different positions. Let's denote these two positions as position 0 and position 1, and think of the dust particle as a two state system:

It could be either of two mutually orthogonal states

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

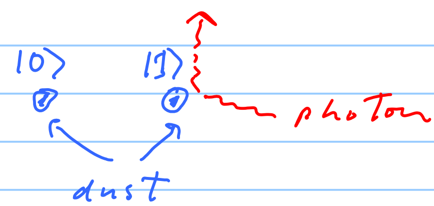
or

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

were we to measure the position, in either of these states, the outcomes 0 and 1 would each occur with probability $\frac{1}{2}$

Yet, the two states $|+\rangle$ and $|-\rangle$ are in principle perfectly distinguishable because the relative phase is different. An interference measurement detects that relative phase.

But, there is other stuff in the room besides the dust particle. Though we have pumped the air out of the room, the vacuum is not perfect -- there are still some gas molecules bouncing around. And though the room is very dark, it is not perfectly dark -- there are a few photons bouncing around as well. A photon, say, might be scattered by the dust particle. And if that happens, the state of the photon after being scattered may depend on whether the position of the dust particle was position 0 or position 1.




To model this situation, let's imagine that the probability of the photon being scattered is $p$. If unscattered its state is $|un\rangle$, but if it is scattered, the state of the photon becomes photon state $|0\rangle$ if the particle is at position 0 and photon state $|1\rangle$ if the particle is at position 1. Under this interaction with the photon, then, the particle states $|0\rangle_{part}$ and $|1\rangle_{part}$ evolve as

$$|0\rangle_S \otimes |un\rangle_E \longmapsto \sqrt{1-p}\; |0\rangle_S \otimes |un\rangle_E + \sqrt{p}\; |0\rangle_S \otimes |0\rangle_E$$

$$|1\rangle_S \otimes |un\rangle_E \longmapsto \sqrt{1-p}\; |1\rangle_S \otimes |un\rangle_E + \sqrt{p}\; |1\rangle_S \otimes |1\rangle_E$$

I'm using the subscript $S$ to denote the system (the dust particle) and the subscript $E$ to denote its environment (the photon). The three states $|un\rangle$, $|0\rangle$, $|1\rangle$ are assumed to be mutually orthogonal states of the environment. The evolution of the joint system $SE$ is <u>unitary</u> — the orthonormal basis states $|0\rangle_S \otimes |un\rangle_E$ and $|1\rangle_S \otimes |un\rangle_E$ evolve to new orthonormal basis states.

To be a bit more general, suppose the state of the dust particle, before interacting with its environment, is

$$|\psi\rangle = a|0\rangle_S + b|1\rangle_S .$$

The density operator of this pure state is

$$\hat{\rho} = |\psi\rangle\langle\psi| = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a^* & b^* \end{pmatrix} = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}$$

After interaction with the environment, the joint state of $SE$ becomes

$$\sqrt{1-p}\, \left( a|0\rangle_S + b|1\rangle_S \right) \otimes |un\rangle_E + \sqrt{p}\; a|0\rangle_S \otimes |0\rangle_E + \sqrt{p}\; b|1\rangle_S \otimes |1\rangle_E$$

The new density operator $\hat{\rho}'$, then, is realized as an ensemble:

$|\psi\rangle$, prob $= 1-p$

$|0\rangle$, prob $= p|a|^2$

$|1\rangle$, prob $= p|b|^2$

or $\rho' = (1-p)\rho + p|a|^2|0\rangle\langle 0| + p|b|^2|1\rangle\langle 1|$

$$= (1-p)\begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} + p\begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} = \begin{pmatrix} |a|^2 & (1-p)ab^* \\ (1-p)a^*b & |b|^2 \end{pmatrix}.$$

The "on-diagonal" entries of the density operator are unaffected by the interaction, but the "off-diagonal" entries are suppressed by the factor $1-p$, the probability that the photon is <u>not</u> scattered.

Suppose that photon scattering occurs at a rate $\Gamma$. This means that the probability of a scattering event in a small time interval $\Delta t$ is

prob(scatt, $\Delta t$) = $\Gamma \Delta t$, and the probability of no scattering is prob(no scatt, $\Delta t$) = $1 - \Gamma\Delta t$

The prob that no scattering event occurs in a finite time interval $t$ is the prob that no scattering occurs in each of $t/\Delta t$ consecutive intervals of width $\Delta t$, or

$$\text{prob(no scatt}, t) = (1-\Gamma\Delta t)^{t/\Delta t} \longrightarrow e^{-\Gamma t}$$

(in the limit $\Delta t \to 0$). Therefore, if scattering occurs at rate $\Gamma$, then in time $t$ the density operator evolves as

$$\hat{\rho}(0) = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \longrightarrow \hat{\rho}(t) = \begin{pmatrix} \rho_{00} & e^{-\Gamma t}\rho_{01} \\ e^{-\Gamma t}\rho_{10} & \rho_{11} \end{pmatrix}$$

This evolution applies even if the initial state is mixed: $\hat{\rho}(0) = \sum_a p_a |\psi_a\rangle\langle\psi_a|$

since each pure state $|\psi_a\rangle\langle\psi_a|$ in the mixture evolves in the same way.

This evolution law for the qubit density operator is called <u>phase damping</u>. If at time $t$ we measure the position of the operator in the $|0\rangle, |1\rangle$ basis, the probability of the outcome $|0\rangle$ or $|1\rangle$ is

$$\text{Prob}(0) = \text{tr}\, |0\rangle\langle 0| \hat{\rho} = \langle 0|\hat{\rho}|0\rangle = \rho_{00}$$

$$\text{Prob}(1) = \text{tr}\, |1\rangle\langle 1| \hat{\rho} = \langle 1|\hat{\rho}|1\rangle = \rho_{11}$$

which is the same as the probability at time 0. These on-diagonal entries in $\hat{\rho}$ are unchanged by phase damping.

But if we measure in the basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$?

Note that
$$|+\rangle\langle +| = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1\ 1) = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$
$$|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(1\ -1) = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

Thus
$$\text{Prob}(+) = \text{tr}\, \hat{\rho}\, |+\rangle\langle +| = \frac{1}{2}(\rho_{00} + \rho_{11} + \rho_{01} + \rho_{10})$$
$$\text{Prob}(-) = \text{tr}\, \hat{\rho}\, |-\rangle\langle -| = \frac{1}{2}(\rho_{00} + \rho_{11} - \rho_{01} - \rho_{10})$$

Of course, $\rho_{00} + \rho_{11} = \text{tr}\,\hat{\rho} = 1$, and
$$\text{Prob}(\pm) = \frac{1}{2} \pm \frac{1}{2} e^{-\Gamma t}(\rho_{01} + \rho_{10}).$$

No matter what the initial state is, after time $t$ such that $\Gamma t \gg 1$ (i.e. $t$ sufficiently late so that scattering has very likely occured), the $+$ and $-$ outcomes are equiprobable. The "phase information" that distinguishes these two possible states has been lost. This is <u>decoherence</u>. In effect, the scattered photon has <u>measured</u> the particle, destroying the potential interference pattern.

Now we can begin to appreciate why superpositions of macroscopically distinguishable states are very fragile. Scattering of just a single photon or air molecule is enough to drive decoherence and destroy interference. For dust particles, even under very well controlled conditions, that can happen really <u>fast</u>.

For microscopic systems the time scale for decoherence is much longer than for macroscopic systems, but it is not infinite. Consider again our model of a two-level atom with Hamiltonian

$$H = E_g |g\rangle\langle g| + E_e |e\rangle\langle e|, \quad \text{where} \quad E_e - E_g = \hbar\omega$$

so that the initial state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \quad \text{evolves to} \quad |\psi(t)\rangle = \frac{1}{\sqrt{2}}(|g\rangle + e^{-i\omega t}|e\rangle)$$

or $$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \frac{1}{2}\begin{pmatrix} 1 \\ e^{-i\omega t}\end{pmatrix}\begin{pmatrix} 1 & e^{i\omega t}\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & e^{i\omega t} \\ e^{-i\omega t} & 1\end{pmatrix}.$$

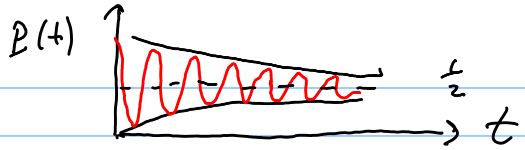
But even a single atom interacts with its environment; not necessarily due to photon scattering in this case, but via other processes that induce a correlation between the state of the environment and whether the state is in the state $|g\rangle$ or $|e\rangle$. Hence the off-diagonal terms in $\hat{\rho}$ decay, so it evolves as

$$\hat{\rho}(t) = \frac{1}{2}\begin{pmatrix} 1 & e^{-\Gamma t}e^{i\omega t} \\ e^{-\Gamma t}e^{-i\omega t} & 1\end{pmatrix}.$$

If we measure the state of the atom in the $|+\rangle, |-\rangle$ basis, we find

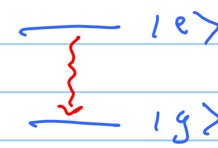
$$P(+) = \langle +|\hat{\rho}|+\rangle = \frac{1}{4}\begin{pmatrix} 1 & 1\end{pmatrix}\begin{pmatrix} 1 & e^{-\Gamma t}e^{i\omega t} \\ e^{-\Gamma t}e^{-i\omega t} & 1\end{pmatrix}\begin{pmatrix} 1 \\ 1\end{pmatrix}$$

$$= \frac{1}{4}\left(2 + e^{-\Gamma t}(e^{i\omega t} + e^{-i\omega t})\right) = \frac{1}{2} + \frac{1}{2}e^{-\Gamma t}\cos\omega t$$




The coherent oscillations are damped by decoherence. A "good qubit" has $\Gamma/\omega \ll 1$, so many oscillations can be observed.

<u>Spontaneous decay</u>




Another type of decoherence can occur in the two-level atom: spontaneous decay of the excited state $|e\rangle$ to the ground state $|g\rangle$, accompanied by the emission of a single photon. When a photon is emitted, the state of the environment changes from the initial zero-photon state $|0\rangle$ to the one-photon state $|1\rangle$. If the excited state decays to the ground state with probability $p$ (while of course the ground state does not decay), the state evolves according to

$$|g\rangle \otimes |0\rangle \longrightarrow |g\rangle \otimes |0\rangle$$

$$|e\rangle \otimes |0\rangle \longrightarrow \sqrt{1-p}\, |e\rangle \otimes |0\rangle + \sqrt{p}\, |g\rangle \otimes |1\rangle .$$

The atomic superposition state evolves as

$$(a|g\rangle + b|e\rangle) \otimes |0\rangle \longrightarrow (a|g\rangle + \sqrt{1-p}\, b|e\rangle) \otimes |0\rangle + \sqrt{p}\, b|g\rangle \otimes |1\rangle$$

and so the density operator of the atom becomes

$$\hat{\rho}' = (a|g\rangle + \sqrt{1-p}\, b|e\rangle)(a^*\langle g| + \sqrt{1-p}\, b^*\langle e|) + p|b|^2 |g\rangle\langle g|$$

$$= \begin{pmatrix} |a|^2 & \sqrt{1-p}\, a b^* \\ \sqrt{1-p}\, a^* b & (1-p)|b|^2 \end{pmatrix} + \begin{pmatrix} p|b|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} |a|^2 + p|b|^2 & \sqrt{1-p}\, a b^* \\ \sqrt{1-p}\, a^* b & (1-p)|b|^2 \end{pmatrix},$$ or in general

$$\hat{\rho} \to \hat{\rho}' = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & (1-p)\,\rho_{11} \end{pmatrix}.$$

Now both the off-diagonal entry and the lower-right diagonal entry are suppressed. If $\Gamma$ is the decay rate of the excited state, then

$1-p = e^{-\Gamma t}$, $\sqrt{1-p} = e^{-\Gamma t/2}$; thus the prob. of being in the excited state decays as

$$\rho_{11} \to e^{-\Gamma t}\rho_{11},$$

while the off-diagonal entries decay as

$$\rho_{\text{off-diag}} \to e^{-\Gamma t/2}\rho_{\text{off-diag}}.$$

For a two level atom undergoing spontaneous decay, then, the coherently oscillating state $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle + e^{-i\omega t}|e\rangle\right)$

or $\hat{\rho}(t) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\omega t} \\ e^{-i\omega t} & 1 \end{pmatrix}$ is damped to become

$$\hat{\rho}'(t) = \begin{pmatrix} 1 - \frac{1}{2}e^{-\Gamma t} & \frac{1}{2}e^{-\Gamma t/2}e^{i\omega t} \\ \frac{1}{2}e^{-\Gamma t/2}e^{-i\omega t} & \frac{1}{2}e^{-\Gamma t} \end{pmatrix}$$

and the probability for the $|+\rangle$ outcome when we measure in the $|+\rangle, |-\rangle$ basis is

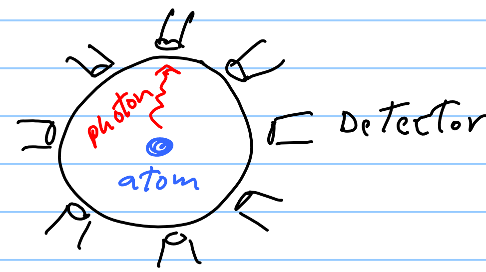
$$P(t) = \langle + | \hat{\rho}'(t) | + \rangle = \frac{1}{2} + \frac{1}{2}e^{-\Gamma t/2}\cos\omega t.$$

In standard nomenclature, $T_1$ denotes the exponential decay time of the excited state, and $T_2$ denotes the exponential decay time for the off-diagonal entries in the density operator. It is often the case in practice that phase damping is much faster than spontaneous decay, so that $T_2 \ll T_1$.

But we see that for the case of pure spontaneous decay the two time scales are related:

$$T_2 = 2/\Gamma = 2T_1 .$$

Decoherence occurs because the state of the atom becomes correlated with state of the environment (in this case the electromagnetic field, which has either zero or one photon). So far we have described how the atom evolves if the state of the environment is not observed. But what if the atom is surrounded by photon detectors, so we <u>know</u> whether or not the atom has emitted a photon. How then does the atom evolve?




Let's go back to the joint pure state of atom and photon:

$$\left(a|g\rangle + \sqrt{1-p}\, b|e\rangle\right) \otimes |0\rangle + \sqrt{p}\, b|g\rangle \otimes |1\rangle .$$

If we detect the photon, then we know the environment is in the state $|1\rangle$, and therefore the state of the atom is $|g\rangle$. But if we don't detect the photon (and our photon detectors are perfectly efficient, so we would have detected it if it had been emitted), then the state of the atom is

$$\frac{a|g\rangle + \sqrt{1-p}\, b|e\rangle}{\| a|g\rangle + \sqrt{1-p}\, b|e\rangle \|}$$

where the norm in the denominator is

$$\| \cdot \|^2 = |a|^2 + (1-p)|b|^2 = 1 - p|b|^2.$$

Substituting again $(1-p) = e^{-\Gamma t}$, we express the state of the atom, when no photon is detected, as

$$|\psi(t)\rangle_{\text{no photon}} = \frac{a|g\rangle + e^{-\Gamma t/2} b|e\rangle}{\sqrt{|a|^2 + e^{-\Gamma t}|b|^2}}.$$

Therefore, even when "nothing happens" (no photon is emitted), the state of the atom evolves, closely approaching the ground state $|g\rangle$ as $\Gamma t \to \infty$.

It is clear why the state of the atom becomes the ground state when a photon is emitted, but why should it become the ground state even when no photon is emitted (after a long wait)? If we start out with the atom in the state $a|g\rangle + b|e\rangle$ at time zero, and we immediately measure in the basis $|g\rangle, |e\rangle$, we would find outcome $|g\rangle$ with probability $|a|^2$ and outcome $|e\rangle$ with probability $|b|^2$. But as time goes by and no photon is observed, we need to adjust this probability distribution a posteriori. Since we have seen no photon it is becoming increasingly likely that the state of the atom was $|g\rangle$ "to begin with." After waiting awhile and seeing "nothing", we should update our description of the atom's state.

As $\Gamma t \to \infty$, the photon will be emitted eventually with probability $|b|^2$, and no photon will ever be emitted with probability $|a|^2$.

In effect, just by waiting, and by noting the response of the photon detector, we measure the atom in the basis $|g\rangle, |e\rangle$ (as $\Gamma t \to \infty$).
Because the system and the environment are correlated, we need to update our description of the system when we observe that the state of its environment does <u>not</u> change.